On the Lattice of Varieties of De Morgan Monoids

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De Morgan monoids

A De Morgan monoid $\mathbf{A} = \langle A; \vee, \wedge, \cdot, \neg, t \rangle$ comprises

- a distributive lattice $\langle A; \vee, \wedge \rangle$
- a square-increasing ($x \leq x \cdot x$) commutative monoid $\langle A; \cdot, t \rangle$
- $x = \neg\neg x$
- $x \cdot y \leq z$ iff $x \cdot \neg z \leq \neg y$
- $x \to y := \neg(x \cdot \neg y)$.

$$\begin{aligned} \mathcal{DM} &:= \{\text{all De Morgan monoids}\} \\ \mathcal{RA} &:= \{t\text{-free subreducts of De Morgan monoids}\} \\ &= \{\text{Subalgebras of } \langle A, \vee, \wedge, \cdot, \neg \rangle \text{ where } \mathbf{A} \in \mathcal{DM}\} \end{aligned}$$

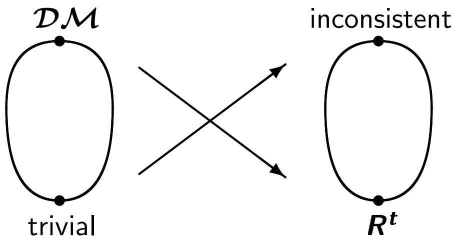
$\mathcal{DM}$ and $\mathcal{RA}$ are varieties.

Algebraic Logic

Define a logic $\boldsymbol{R^t}$ as follows

$$\gamma_1, \ldots, \gamma_n \vdash_{\boldsymbol{R^t}} \alpha \text{ iff } \mathcal{DM} \models (t \leq \gamma_1 \,\&\, \ldots \,\&\, t \leq \gamma_n) \implies t \leq \alpha.$$

Similarly for $\mathcal{RA}$ and the logic $\boldsymbol{R}$, where we replace every $t \leq \alpha$ with $\alpha \to \alpha \leq \alpha$, to which it is equivalent in $\mathcal{DM}$.

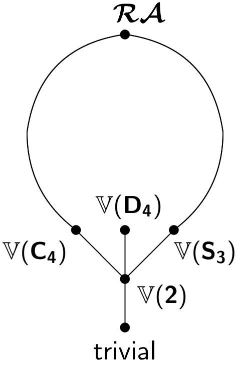


Subvarieties of $\mathcal{DM}$

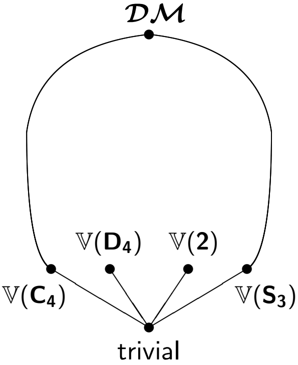
Axiomatic extensions of $\boldsymbol{R^t}$

$\mathcal{DM}$ vs $\mathcal{RA}$

- $\boldsymbol{R}$ and $\boldsymbol{R^t}$ have the same rules/theorems not involving t.
- Every finitely generated algebra in $\mathcal{RA}$ has a unique identity element for $\cdot$ and is therefore a reduct of a De Morgan monoid.

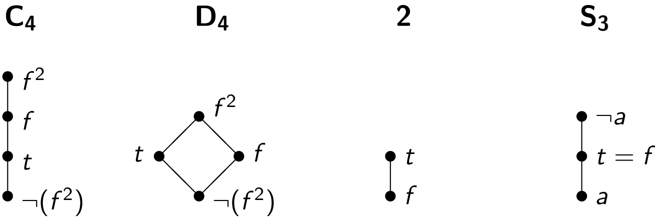


Subvarieties of $\mathcal{RA}$
(Świrydowicz 1995)



Subvarieties of $\mathcal{DM}$

Important Algebras



$$f := \neg t$$

These are all simple (which amounts to t having just one strict lower bound)

Structural Completeness

- Raftery and Świrydowicz (2016) showed recently that the only non-trivial (passively) structurally complete subvariety of $\mathcal{RA}$ is the variety of Boolean algebras.
- Which subvarieties of $\mathcal{DM}$ are structurally complete?

- A variety $\mathcal{V}$ called *structurally complete* if every proper subquasivariety of $\mathcal{V}$ generates a proper subvariety of $\mathcal{V}$.
- $\mathcal{V}$ is called *passively structurally complete* if all the non-trivial algebras in $\mathcal{V}$ satisfy the same existential positive sentences.

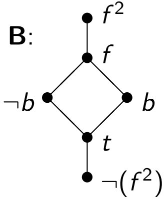
Passive Structural Completeness in $\mathcal{DM}$

Thm. *A variety $\mathcal{K} \subseteq \mathcal{DM}$ is passively structurally complete iff one of the following four (mutually exclusive) conditions hold:*

1. $\mathcal{K} = \mathbb{V}(\mathbf{2})$;
2. $\mathcal{K} = \mathbb{V}(\mathbf{D_4})$;
3. $\mathcal{K}$ *consists of odd Sugihara monoids;*
4. $\mathbf{C_4}$ *is a retract of every non-trivial algebra in* $\mathcal{K}$.

The class $\{\mathbf{A} \in \mathcal{DM} : \mathbf{A} \text{ is trivial or } \mathbf{C_4} \text{ is a retract of } \mathbf{A}\}$ is a quasivariety but not a variety. For example $\mathbf{C_4}$ is a retract of $\mathbf{B} \times \mathbf{C_4}$, but $\mathbf{B}$ is a *simple* homomorphic image of $\mathbf{B} \times \mathbf{C_4}$ and so can't map onto $\mathbf{C_4}$.

B:

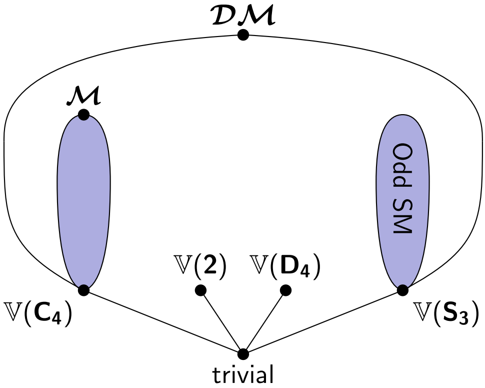


with $b \cdot b = f^2$, $\neg b \cdot \neg b = f^2$ and $b \cdot \neg b = f$.

Exploring condition 4

Thm. *There is a largest subvariety $\mathcal{M}$ of $\mathcal{DM}$ such that $\mathbf{C_4}$ is a retract of all non-trivial members of $\mathcal{M}$. $\mathcal{M}$ is axiomatised, relative to $\mathcal{DM}$, by:*

- $t \le f$,
- $x \le f^2$,
- $((f \to x) \vee (x \to t)) \to 0 = 0 \quad [0 := \neg(f^2)]$.



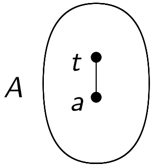
Exploring $\mathcal{M}$

Every subdirectly irreducible algebra in $\mathcal{M}$ arises by a construction of J. K. Slaney (1993) from a **Dunn monoid A** [essentially a De Morgan monoid without negation], i.e.,

> *a square-increasing distributive lattice-ordered commutative monoid* $\langle A; \vee, \wedge, \cdot, \rightarrow, t \rangle$ *that satisfies the law of residuation*
>
> $$x \leq y \rightarrow z \text{ iff } x \cdot y \leq z.$$

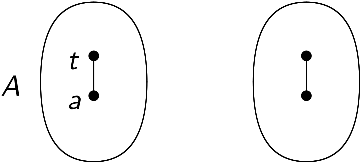
Let's call this construction **skew reflection**.

Skew Reflection



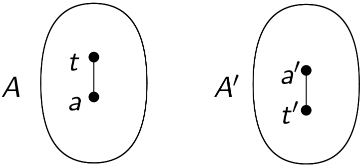
Dunn monoid

Skew Reflection

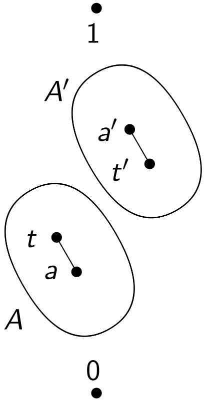


Dunn monoid

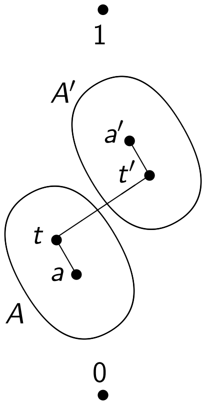
Skew Reflection



Skew Reflection



Skew Reflection



Declare that $a < b'$ for certain $a, b \in A$ in such a way that $\langle A \cup A' \cup \{0, 1\}; \leq \rangle$ is a distributive lattice, $t < t'$ and for all $a, b, c \in A$,

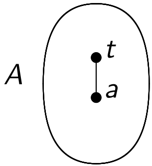
$$a \cdot b < c' \text{ iff } a < (b \cdot c)'.$$

Then there is a unique way of turning the structure into a De Morgan monoid

$$S^{<}(\mathbf{A}) = \langle A \cup A' \cup \{0, 1\}; \vee, \wedge, \cdot, \neg, t \rangle \in \boldsymbol{\mathcal{M}},$$

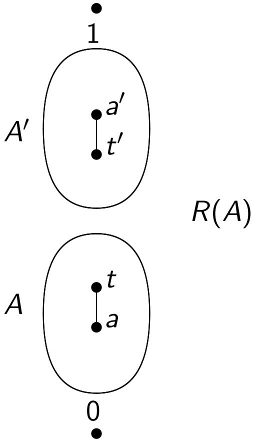
of which $\mathbf{A}$ is a subreduct, where $\neg$ extends $'$. In particular if we specify that $a < b'$ for all $a, b \in A$, then we get the **reflection** construction, which is an older idea, see Meyer (1973) and Galatos and Raftery (2004). In this case we write $R(\mathbf{A})$.

Reflection



Dunn monoid

Reflection



Consequences of Reflection

- Adding an $\boldsymbol{\mathcal{M}}$-style negation to Dunn monoids makes no difference to the negation-less equational theory.
- The equational theory of $\boldsymbol{\mathcal{M}}$ is undecidable. (cf., Urquhart 1984)

Recall Q: Which subvarieties of $\boldsymbol{\mathcal{M}}$ are structurally complete?

The map

$$\boldsymbol{\mathcal{W}} \mapsto \mathbb{V}\{R(\mathbf{A}) : \mathbf{A} \in \boldsymbol{\mathcal{W}}\},$$

from varieties of Dunn monoids to subvarieties of $\boldsymbol{\mathcal{M}}$, preserves structural *incompleteness*. Therefore some subvarieties of $\boldsymbol{\mathcal{M}}$ are *not* structurally complete e.g.

$$\mathbb{V}\{R(\mathbf{A}) : \mathbf{A} \text{ a Brouwerian algebra}\}$$

Skew Reflection

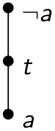
If we subject the algebras $\mathbf{2}$, $\mathbf{S_3}$, $\mathbf{C_4}$ and $\mathbf{D_4}$ to all possible executions of skew reflection within $\boldsymbol{\mathcal{M}}$, then we get four algebras that generate covers of $\mathbb{V}(\mathbf{C_4})$, viz.

$$R(\mathbf{2}), R(\mathbf{S_3}), S^{<}(\mathbf{S_3}), S^{<}(\mathbf{C_4}).$$

We suspect that these are the only covers of $\mathbb{V}(\mathbf{C_4})$ within $\boldsymbol{\mathcal{M}}$ (work in progress). Let's exhibit some of the generating algebras.

Constructing $R(\mathbf{S_3})$

$\mathbf{S_3}$:



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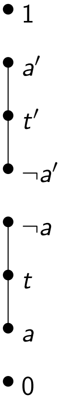
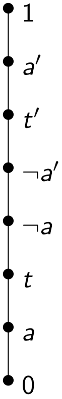
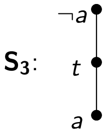


Constructing $R(\mathbf{S_3})$

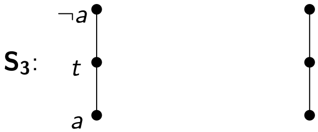
$R(\mathbf{S_3})$:



Constructing $S^{<}(\mathbf{S_3})$



Constructing $S^{<}(\mathbf{S_3})$



Constructing $S^{<}(\mathbf{S_3})$

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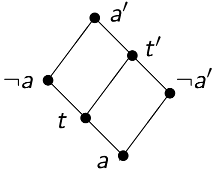
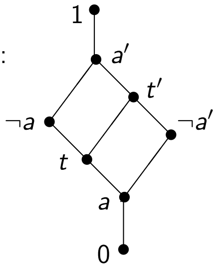
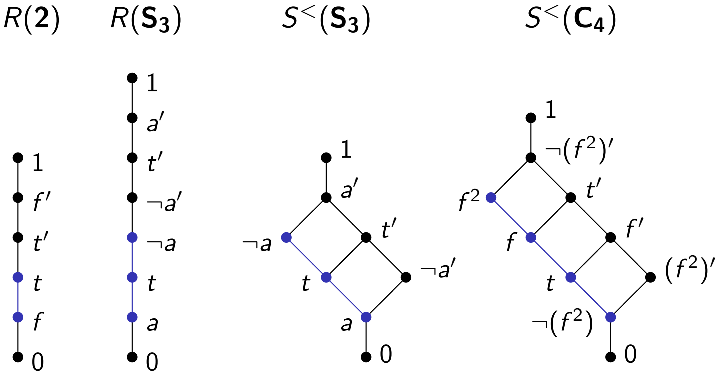


Constructing $S^<(\mathbf{S_3})$

$S^<(\mathbf{A})$:



Covers of $\mathbb{V}(\mathbf{C_4})$



Structurally Complete Subvarieties of $\mathcal{M}$

We have the following positive results:

Thm. *The subvarieties* $\mathbb{V}(R(\mathbf{2}))$, $\mathbb{V}(R(\mathbf{S_3}))$, $\mathbb{V}(S^{<}(\mathbf{C_4}))$ *and* $\mathbb{V}(S^{<}(\mathbf{S_3}))$ *are structurally complete.*

Here $\overline{\mathbf{A}}$ denotes $\mathbb{V}(\mathbf{A})$, and S is really $S^{<}$.