Undecidability of some modal product logics

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June 28, 2016

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Modal product logics

Language: $\&, \rightarrow, 0$ plus two unary (modal) symbols $(\Box, \Diamond)$

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Definition

A (crisp) product Kripke model $\mathfrak{M}$ is a tripla $\langle W, R, e\rangle$ where:

- $R \subseteq W \times W$ (Rus stands for $\langle u, s\rangle \in R$)
- $e : W \times Var \rightarrow [0, 1]$ uniquelly extended by:
  - $e(u, \varphi \& \psi) = e(u, \varphi) \cdot e(u, \psi)$,
  - $e(u, \varphi \rightarrow \psi) = \begin{cases} 1 & \text{if } e(u, \varphi) \leq e(u, psi) \\ e(u, \psi)/e(u, \varphi) & \text{otherwise} \end{cases}$,
  - $e(u, \Box\varphi) = inf\{e(s, \varphi) : Rus\}$
  - $e(u, \Diamond\varphi) = sup\{e(s, \varphi) : Rus\}$

Modal product logics

- **(Global deduction)**: $\Gamma \Vdash \varphi$ iff $[\forall u \in W \ e(u, [\Gamma]) \subseteq \{1\}]$ implies $[\forall u \in W \ e(u, \varphi) = 1]$ for all product Kripke models $\mathfrak{M}$.
  $\Gamma \Vdash^f \varphi$ for denoting the same relation over finite product Kripke models.

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$\Gamma \Vdash^{f} \varphi$ for denoting the same relation over finite product Kripke models.
- **(Local deduction)**: $\Gamma \vdash_4 \varphi$ iff
$\forall u \in W \ [e(u, [\Gamma]) \subseteq \{1\}$ implies $e(u, \varphi) = 1]$ for all **transitive** product Kripke models $\mathfrak{M}$.
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Post Correspondence Problem

An instance of the PCP is a list of pairs $\langle \mathbf{v_1}, \mathbf{w_1} \rangle \ldots \langle \mathbf{v_n}, \mathbf{w_n} \rangle$ where $\mathbf{v_i}, \mathbf{w_i}$ are numbers in base $s \geq 2$.

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It is undecidable whether there exist $i_1, \ldots, i_k$ such that

$$\mathbf{v_{i_1}} \cdots \mathbf{v_{i_k}} = \mathbf{w_{i_1}} \cdots \mathbf{w_{i_k}}$$

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- $\mathbf{a}, \mathbf{b}$ numbers in base $s \implies \mathbf{ab} = \mathbf{a} \cdot s^{\|\mathbf{b}\|} + \mathbf{b}$, where $\| \mathbf{b} \|$ is the length of $\mathbf{b}$ (in base s).

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- we can exploit the product operation to express concatenation (using powers over $x \in (0, 1)$)

The global modal logic case

Given a PCP instance P there is a finite set of formulas $\mathit{\Gamma}_g(P) \cup \{\varphi_g\}$ such that

$$P \text{ is SAT} \iff \mathit{\Gamma}_g(P) \not\Vdash \varphi_g$$

Moreover $\mathit{\Gamma}_g(P) \Vdash \varphi_g \iff \mathit{\Gamma}_g(P) \Vdash^f \varphi_g$.

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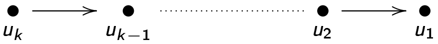
- Proving $\Longrightarrow$ will not be hard (constructing a model using the solution of P).
- Idea for $\Longleftarrow$: if $\Gamma_g(P) \not\Vdash \varphi_g$ then it happens in u_k of a particular structure shaped like



The global case: formulas

Variables used: $\mathcal{V} = \{x, y, z, v, w\}$. y, z, are control variables; x stores information on the index of the added word; v, w store information on the concatenation.

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- $(\neg\Box\bar{0}) \rightarrow (\Box p \leftrightarrow \Diamond p)$ for each $p \in \mathcal{V}$:

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Lemma

If $\Gamma_g(P) \not\Vdash \psi$ (for arbitrary ψ in $\mathcal{V}$) then there is a product Kripke model $\mathfrak{M}$ with $W = \{u_i : i \in \omega\}$ or $W = \{u_i : i \leq k\}$ and $R = \{\langle u_i, u_{i+1}\rangle\}$ such that

- $\mathfrak{M}$ is a model for $\Gamma_g(P)$ and
- $e(u_1, \psi) < 1$

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- $\neg\Box\overline{0} \to ((y \leftrightarrow \Box y) \wedge (z \leftrightarrow \Box z))$: y and z take constant values (if Rut then $e(u, y) = e(t, y) = \alpha_y$ and $e(u, z) = e(t, z) = \alpha_z$)

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- $(x \leftrightarrow z^i) \rightarrow (v \leftrightarrow (\Box v)^{s^{\|\mathbf{v}_i\|}} \& y^{\mathbf{v}_i})$ for $1 \leq i \leq n$:

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Let $\varphi_g = (v \leftrightarrow w) \to (z \vee y)$. For a model $\mathfrak{M}$ satisfying $\Gamma_g(P)$ and with $e(u_k, \varphi_g) < 1$, it is immediate that $\alpha_y, \alpha_z < 1$. (enough for using the power over them).

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The global case: formulas

Lemma

Let $\mathfrak{M}$ with $W = \{u_i : 1 \leq i \leq k\}$ and $R = \{\langle u_{i+1}, u_i \rangle : 1 \leq i < k\}$ be a model of $\Gamma_g(P)$ such that $e(u_k, \varphi_g) < 1$. Then

1. for all $1 \leq j \leq k$

$$e(u_j, v) = \alpha_y^{v_{i_1} \cdots v_{i_j}} \qquad \text{and} \qquad e(u_j, w) = \alpha_y^{w_{i_1} \cdots w_{i_j}}$$

for $e(u_j, x) = \alpha_z^{i_j}$ for $1 \leq j \leq k$.

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1. (inverse) induction on j.
2.
   - $e(u_k, v) = \alpha_y^{m_1}$ and $e(u_2, w) = \alpha_y^{m_2}$ for some integers m_1, m_2
   - $e(u_k, v \leftrightarrow w) = \alpha_y^{|m_1 - m_2|}$, so if $m_1 \neq m_2$ it holds that $e(u_k, v \leftrightarrow w) \leq \alpha_y$!

From P to a model and back

- If $i_1, \ldots, i_k$ is a solution for P, then $\Gamma_g(P) \not\Vdash^{(f)} \varphi_g$ in u_k of the model $\mathfrak{M} = \langle \{u_1, \ldots, u_k\}, \{\langle u_k, u_{k-1}\rangle, ..., \langle u_2, u_1\rangle\}, e\rangle$ with

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  - $e(u_i, z) = \alpha_z > 0$ but small enough:
    $\leq \min_{1 \leq j \leq k} \{ \min_{1 \leq r \leq n} \{e(u_j, v) \leftrightarrow e(u_j, \Box v)^{s^{\|\mathbf{v}_r\|}} \cdot \alpha_y^{\mathbf{v}_r}\}\} \wedge$
    $\min_{1 \leq j \leq k} \{ \min_{1 \leq r \leq n} \{e(u_j, w) \leftrightarrow e(u_j, \Box w)^{s^{\|\mathbf{w}_r\|}} \cdot \alpha_y^{\mathbf{w}_r}\}\}$

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  - $e(u_j, x) = \alpha_z^{i_j}$

From P to a model and back

- If $i_1, \ldots, i_k$ is a solution for P, then $\Gamma_g(P) \not\Vdash^{(f)} \varphi_g$ in u_k of the model $\mathfrak{M} = \langle \{u_1, \ldots, u_k\}, \{\langle u_k, u_{k-1}\rangle, \ldots, \langle u_2, u_1\rangle\}, e\rangle$ with
  - $e(u, y) = \alpha_y \in (0, 1)$
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  - $e(u_j, x) = \alpha_z^{i_j}$ observe $e(u_j, x \leftrightarrow z^r)$ for $1 \leq r \leq n$ is either 1 (if $r = i_j$) or is $\leq \alpha_z$.

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- Similarly, if $\Gamma_g(P) \not\Vdash^{(f)} \varphi_g$ in u_k of a model $\mathfrak{M}$ as the ones presented we can extract easily a solution for P.

The local modal logic case

In a similar fashion as before we can define a finite set $\mathit{\Gamma}_l(P) \cup \{\varphi_l\}$ (in the same $\mathcal{V}$) such that

$$P \text{ is SAT} \iff \mathit{\Gamma}_l(P) \nvdash_4 \varphi_l$$

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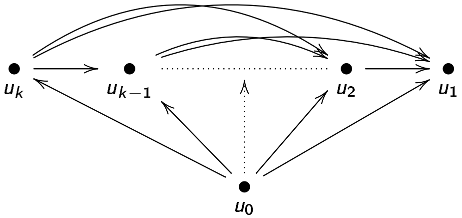
We now work towards structures with the form



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- $\Box(\Box(v \& w) \to (\Box v \& \Box w))$: ensures the witness of $\Box v$ and $\Box w$ coincides.

The φ_l formula is the same as in the global case but again closed under $\Box$.

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The construction of a model $\mathfrak{M}$ from a solution of P and viceversa are similar to the ones from the global case.

Thank you!