Undecidability of some Product Modal Logics

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Fuzzy modal logic is a field that has received interest in the last years. In general, there are several ways to expand a concrete fuzzy logic with modal operators. This is mainly due to two reasons. First, in fuzzy modal logic the modal operators $\Box$ and $\Diamond$ are not in general inter-definable and, therefore, it makes sense to consider modal expansions of a given fuzzy logic either with $\Box$ or with $\Diamond$ or both with $\Box$ and $\Diamond$. Second, the relational semantics for fuzzy modal logics can be defined in terms of crisp or fuzzy accessibility relation and this gives rise to different modal logics over the same fuzzy logic. Special attention has been devoted to modal expansion of the fuzzy logics associated with the basic continuous t-norms: Łukasiewicz modal logics [9, 8], Gödel modal logics [4, 5] and Product modal logics [11]. For further information on modal expansions of t-norm based logics see [2, 10].

The study of decision problems in fuzzy modal logic has focused on Gödel-style logics [3] and fuzzy description logics [1, 6, 7]. In the current contribution we present two undecidability results of modal Product logic. Let $\vdash^g$ be the *global* consequence relation of Kripke models with a crisp accessibility relation evaluated over the standard Product algebra expanded with the Monteiro-Baaz Δ operator, considering both $\Box$ and $\Diamond$ modal operators. Let $\vdash_4$ be the *local* consequence relation of the transitive models of the previous class.

Lemma

1. *Determining whether a formula φ is a $\vdash^g$-consequence of a finite set Γ is undecidable.*
2. *Determining whether a formula φ is a $\vdash_4$-consequence of a finite set Γ is undecidable.*

In the path to prove the previous lemma we also prove that the same questions over the logics (denoted respectively by $\vdash^g_f$ and $\vdash_{f4}$) determined by the finite models of the corresponding class are also undecidable. Moreover, since the local modal logic enjoys the Δ-deduction theorem, from *2*. we get the following corollary (and their corresponding version over $\vdash_{f4}$).

Corollary

1. *Determining whether a formula φ is a theorem of $\vdash_4$ is undecidable.*

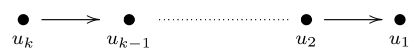

Figure 1: Frame for the Global logic proof

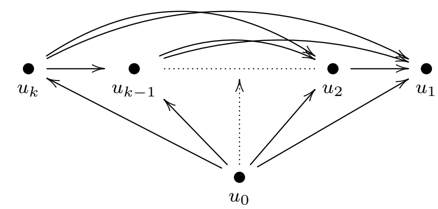

Figure 2: Frame for the Local logic proof

2. *Determining whether a formula φ is 1-satisfiable in $\vdash_4$ is undecidable.*

The undecidability results are done via reduction of the Post correspondence problem (Pcp) to the ones above. An instance of the Pcp is a list $\{\langle \mathtt{v_1}, \mathtt{w_1}\rangle, \ldots, \langle \mathtt{v_m}, \mathtt{w_m}\rangle\}$ of numbers in base $s \in \mathbb{N}^+$, and a solution for it is a finite list $i_1, \ldots, i_k$ with $i_j \in \{1, \ldots, m\}$ such that $\mathtt{v_{i_1}} \ldots \mathtt{v_{i_k}} = \mathtt{w_{i_1}} \ldots \mathtt{w_{i_k}}$[1]. Given an instance of the Pcp we can define two particularly interesting finite sets of formulas, $\mathscr{G} \cup \{\varphi_G\}$ and $\mathscr{L} \cup \{\varphi_L\}$ using only the variables $\mathcal{V} = \{x, y, v, w\} \cup \{x_i : 1 \leqslant i \leqslant m\}$.

Concerning the global case, it is possible to check that if $\mathscr{G} \not\vdash^g \varphi_G$ then there is a model $\mathfrak{M}$ with the structure of Figure 1 such that:

1. $\mathfrak{M}$ globally satisfies $\mathscr{G}$, $e(u_k, \varphi_G) < 1$, and this implies that $e(u_k, v) = e(u_k, w)$,
2. y takes the same value (α_y) in all worlds of the model,
3. x equals x_j for some $1 \leqslant j \leqslant m$ at each world of the model (and all x_i are different),
4. For $1 \leqslant j \leqslant k$, $e(u_j, v) = \alpha_y^{\mathtt{v_{i_1}} \ldots \mathtt{v_{i_j}}}$, where $i_n = r$ for $e(u_n, x) = e(u_n, x_r)$. The analogous happens for $e(u_j, w)$.

Using the previous results (and given that $\mathfrak{M}$ is finite) we prove the following.

$$P \text{ is satisfiable} \iff \mathscr{G} \not\vdash^g \varphi_G \iff \mathscr{G} \not\vdash^g_f \varphi_G.$$

With some more effort but in a similar fashion, we can prove some undecidability results concerning the local deduction *over transitive models*. We show that if $\mathscr{L} \not\vdash_4 \varphi_L$ then there is a model $\mathfrak{M}$ with the structure of Figure 2 such that $e(u_0, [\mathscr{L}]) = \{1\}$ and $e(u_0, \varphi_L) < 1$ (and again this implies that $e(u_k, v) = e(u_k, w)$), and the other characteristics listed for the global case are preserved.

As before we can prove the following.

$$P \text{ is satisfiable} \iff \mathscr{L} \not\vdash_4 \varphi_L \iff \mathscr{L} \not\vdash_{f4} \varphi_L.$$

Acknowledgments

[1] The concatenation of numbers.

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