A uniform way to build strongly perfect MTL-algebras via Boolean algebras and prelinear semihoops

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MTL-algebras

A prelinear semihoop is an algebra $\mathbf{H} = (H, *, \to, \wedge, \vee, 1)$ such that it is a commutative integral prelinear residuated lattice.

A MTL-algebra is an algebra $\mathbf{A} = (A, *, \to, \wedge, \vee, 0, 1)$ that is a commutative integral pointed prelinear residuated lattice.

Strongly perfect MTL-algebras (*SBP_0-algebra*), are MTL-algebras satisfying:

(N) $\neg(x)^2 \to (\neg\neg x \to x) = 1,$

(DL) $(2x)^2 = 2(x^2).$

Subvarieties of strongly perfect MTL-algebras

SMTL: MTL $+$ $x \wedge \neg x = 0$ Pseudocomplemented MTL-algebras

- Product algebras: BL $+$ $\neg x \vee ((x \to x \cdot y) \to y) = 1$
- Gödel algebras: BL $+$ $x \cdot x = x$

IBP_0 : SBP_0 $+$ $\neg\neg x = x$ Involutive SBP_0-algebras

- DLMV : IBP_0 $+$ $x * (x \to y) = y * (y \to x)$, the variety generated by perfect MV-algebras
- NM^- : IBP_0 $+$ $\neg(x^2) \vee (x \to x^2) = 1$, the variety generated by the nilpotent minimum algebra $[0,1] \backslash \{1/2\}$.

CH-Triples and product algebras

[Montagna - U., 2015]: The category $\mathbb{P}$ of product algebras is equivalent to a category whose objects are triples $(\mathbf{B}, \mathbf{C}, \vee_e)$, where $\mathbf{B}$ is a Boolean algebra, $\mathbf{C}$ is a cancellative hoop and $\vee_e : B \times C \to C$ satisfies suitable properties.

CH-Triples and product algebras

[Montagna - U., 2015]: The category $\mathbb{P}$ of product algebras is equivalent to a category whose objects are triples $(\mathbf{B}, \mathbf{C}, \vee_e)$, where $\mathbf{B}$ is a Boolean algebra, $\mathbf{C}$ is a cancellative hoop and $\vee_e : B \times C \to C$ satisfies suitable properties.

Key idea Directly indecomposable product algebras are of the kind $\mathbf{2} \oplus \mathbf{C}$ [Cignoli, Torrens].

CH-Triples and product algebras

Let $\mathbf{P}$ be a product algebra, then $\mathbf{P} \hookrightarrow \prod_{i \in I} \mathbf{2} \oplus \mathbf{C}_i$ and given any $p \in P$, let p_i be its i-th component. Hence p_i either is Boolean, $p_i \in \{0, 1\}$, or it is cancellative, $p_i \in C_i$.

In particular $p_i = b_i \wedge c_i$, for some $b_i \in B_i, c_i \in C_i$, and hence each $p \in P$

$$p = b \wedge c$$

for $b \in \mathbf{B_P}$, the greatest Boolean subalgebra, or the Boolean skeleton, of $\mathbf{P}$, and $c \in \mathbf{C_P}$, the greatest cancellative subhoop of $\mathbf{P}$.

Decomposition of a product algebra

Let $\mathbf{P} = (P, *, \to, \wedge, \vee, 0, 1)$ be a product algebra. We can uniquely associate to $\mathbf{P}$ the pair $(\mathbf{B_P}, \mathbf{C_P})$:

- $B_P = \{x \in P \,|\, \neg\neg x = x\} = \{x \in P \,|\, x \vee \neg x \ = 1\}$ is the dominium of the greatest Boolean subalgebra of $\mathbf{P}$.
- $C_P = \{x \in P \,|\, \neg\neg x = 1\} = \{x \in P \,|\, x > \neg x\}$ is the dominium of the greatest cancellative subhoop of $\mathbf{P}$. Note that C_P is the radical of $\mathbf{P}$.

But a pair $(\mathbf{B}, \mathbf{C})$ does not uniquely determine $\mathbf{P}$.

CH-triples

A CH-triple is a triple $(\mathbf{B}, \mathbf{C}, \vee_e)$ where $\mathbf{B}$ is a Boolean algebra, $\mathbf{C}$ is a cancellative hoop such that $B \cap C = \{1\}$, and $\vee_e$ is a map from $\mathbf{B} \times \mathbf{C}$ into $\mathbf{C}$ such that:

(V1) For fixed $b \in B$ and $c \in C$:
the map $h_b(x) = b \vee_e x$ is an endomorphism of $\mathbf{C}$,
the map $k_c(x) = x \vee_e c$ is a lattice homomorphism from $\mathbf{B}$ into $\mathbf{C}$.

(V2) h_0 is the identity on $\mathbf{C}$,
h_1 is constantly equal to 1.

(V3) For all $b, b' \in B$ and for all $c, c' \in C$,
$h_b(c) \vee h_{b'}(c') = h_{b \vee b'}(c \vee c') = h_b(h_{b'}(c \vee c'))$.

The category of CH-triples

A good morphism pair from a CH-triple $(\mathbf{B}, \mathbf{C}, \vee_e)$ to another CH-triple $(\mathbf{B}', \mathbf{C}', \vee_e')$ is a pair (h, k) where:

- h is a homomorphism from $\mathbf{B}$ to $\mathbf{B}'$,
- k is a homomorphism from $\mathbf{C}$ to $\mathbf{C}'$,
- for all $x \in B$ and $y \in C$, $k(x \vee_e y) = h(x) \vee_e' k(y)$.

The category $\mathbb{CHT}$ of CH-triples has CH-triples as objects and good morphism pairs as morphisms, with composition defined componentwise: $(h, k) \circ (h', k') = (h \circ h', k \circ k')$.

We define a functor Φ from the category of product algebras $\mathbb{P}$ to $\mathbb{CHT}$ as follows:

- $\Phi(\mathbf{P}) = (\mathbf{B_P}, \mathbf{C_P}, \vee)$.
- $\Phi(f) = (f_{|_{B_P}}, f_{|_{C_P}})$, for any morphism $f : \mathbf{P} \to \mathbf{P}'$.

Inverting Φ: building a product algebra

Let $(\mathbf{B}, \mathbf{C}, \vee_e)$ be a CH-triple. For all $b, b' \in B$, and $c, c' \in C$ let:

$$(b, c) \sim (b', c') \text{ iff } b = b' \text{ and } \neg b \vee_e c = \neg b \vee_e c'$$

We define $\mathbf{B} \otimes_{\vee_e} \mathbf{C} = ((B \times C)/\sim, \otimes, \Rightarrow, \sqcap, \sqcup, [0,1], [1,1])$, where:

$$[b, c] \otimes [b', c'] = [b \wedge b',\ c \cdot c']$$

$$[b, c] \Rightarrow [b', c'] = [b \to b',\ \neg b \vee_e (c \to c')]$$

$$[b, c] \sqcap [b', c'] = [b \wedge b',\ c \wedge c']$$

$$[b, c] \sqcup [b', c'] = [b \vee b', ((\neg b \vee \neg b') \vee_e (c \vee c')) \wedge ((b \vee \neg b') \vee_e c') \wedge ((b' \vee \neg b) \vee_e c)]$$

Theorem

$\mathbf{B} \otimes_{\vee_e} \mathbf{C}$ *is a product algebra.*

Inverting Φ: building a product algebra

We define functor Ξ from $\mathbb{CHT}$ into $\mathbb{P}$ as follows:

- $\Xi(\mathbf{B}, \mathbf{C}, \vee_e) = \mathbf{B} \otimes_{\vee_e} \mathbf{C}$.
- $\Xi(h, k)([b, c]) = [h(b), k(c)]$.

Theorem

(Φ, Ξ) *provide a categorical equivalence between* $\mathbb{P}$ *and* $\mathbb{CHT}$.

DLMV algebras and CH-triples

[Esteva, Gispert, Noguera - 2005] If $\mathbf{A}$ is a directly indecomposable DLMV-algebra, thus an element of the variety generated by perfect MV algebras, then it is a disconnected rotation of a cancellative hoop $\mathbf{C}$: $A = \{0, 1\} \times C$.

Any element $a \in A$ can be expressed by means of a pair (b, c) where b is Boolean, $b \in \{0, 1\}$, and c is cancellative. In particular:

$$a = (b \wedge c) \vee (\neg b \wedge \neg c)$$

From DLMV algebras to CH-triples

Also in this case, we can define functor Φ_D from the category of DLMV-algebras $\mathbb{DLMV}$ into $\mathbb{CHT}$ as follows:

- $\Phi_D(\mathbf{A}) = (\mathbf{B_A}, \mathbf{C_A}, \vee)$.
- $\Phi_D(f) = (f_{|_{B_A}}, f_{|_{C_A}})$, for any morphism $f : \mathbf{A} \to \mathbf{A}'$.

From CH-triples to DLMV-algebras

Let $(\mathbf{B}, \mathbf{C}, \vee_e)$ be a CH-triple, and let $h_b : C \to C$, $h_b(c) = b \vee_e c$ for all $b \in B$ and $c \in C$. We define

$$\mathbf{B} \otimes^D_{\vee_e} \mathbf{C} = (B \times C, \odot, \Rightarrow, \sqcap, \sqcup, [0,1], [1,1])$$

where, for every $(b,c), (b',c') \in B \times C$

$(b,c) \odot (b',c') =$
$(b \wedge b', h_{b \vee b'}(1) \wedge h_{b \vee \neg b'}(c' \to c) \wedge h_{\neg b \vee b'}(c \to c') \wedge h_{\neg b \vee \neg b'}(c * c'))$;

$(b,c) \Rightarrow (b',c') =$
$(b \to b', h_{b \vee b'}(c' \to c) \wedge h_{b \vee \neg b'}(1) \wedge h_{\neg b \vee b'}(c * c') \wedge h_{\neg b \vee \neg b'}(c \to c'))$;

$(b,c) \sqcap (b',c') =$
$(b \wedge b', h_{b \vee b'}(c \vee c') \wedge h_{b \vee \neg b'}(c) \wedge h_{\neg b \vee b'}(c') \wedge h_{\neg b \vee \neg b'}(c \wedge c'))$;

$(b,c) \sqcup (b',c') =$
$(b \vee b', h_{b \vee b'}(c \wedge c') \wedge h_{b \vee \neg b'}(c') \wedge h_{\neg b \vee b'}(c) \wedge h_{\neg b \vee \neg b'}(c \vee c'))$.

$\mathbf{B} \otimes^D_{\vee_e} \mathbf{C}$ is a DLMV-algebra.

From CH-triples to DLMV-algebras

We define functor Ξ_D from $\mathbb{CHT}$ to $\mathbb{DLMV}$ as follows:

- $\Xi_D(\mathbf{B}, \mathbf{C}, \vee_e) = \mathbf{B} \otimes_{\vee_e}^{D} \mathbf{C}$.
- $\Xi_D(h, k)(b, c) = (h(b), k(c))$.

Theorem

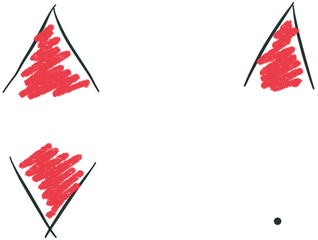
(Φ_D, Ξ_D) *provide a categorical equivalence between* $\mathbb{DLMV}$ *and* $\mathbb{CHT}$.

Corollary

$\mathbb{DLMV}$ *and* $\mathbb{P}$ *are categorically equivalent.*

Generalizing the construction

- Directly indecomposable IBP_0-algebras are disconnected rotations of prelinear semihoops.
- Directly indecomposable SMTL-algebras are liftings of prelinear semihoops.



Generalizing the construction

- We can prove that IBP_0-algebras and SMTL-algebras are categorically equivalent to the category of triples $(\mathbf{B}, \mathbf{H}, \vee_e)$, with $\mathbf{H}$ a prelinear semihoop.

- We can express these results in a uniform manner, and generalize them, using a weakening of Cignoli-Torrens dl-admissible operator δ.

Generalizing the construction

Let $\mathbf{H} = (H, *, \to, \wedge, \vee, 1)$ be a prelinear semihoop. A map $\delta : H \to H$ is dl-admissible iff for all $a, b \in H$:

$$
\begin{array}{rclrcl}
a \to \delta(a) & = & 1, & \delta(\delta(a)) & = & a, \\
\delta(a \to b) & = & a \to \delta(b), & \delta(a * b) & = & \delta(\delta(a) * \delta(b)), \\
\delta(a \wedge b) & = & \delta(a) \wedge \delta(b), & \delta(a \vee b) & = & \delta(a) \vee \delta(b).
\end{array}
$$

Generalizing the construction

Let $\mathbf{H} = (H, *, \to, \wedge, \vee, 1)$ be a prelinear semihoop. A map $\delta : H \to H$ is w-admissible iff for all $a, b \in H$:

$$
\begin{array}{rclrcl}
a \to \delta(a) & = & 1, & \delta(\delta(a)) & = & a, \\
\delta(a \to b) & \leq & a \to \delta(b), & \delta(a * b) & = & \delta(\delta(a) * \delta(b)), \\
\delta(a \wedge b) & = & \delta(a) \wedge \delta(b), & \delta(a \vee b) & = & \delta(a) \vee \delta(b).
\end{array}
$$

Observation: the weakened condition allows to get rid of Glivenko equation $\neg\neg(\neg\neg x \to x) = 1$.

Generalizing the construction

Let $\mathbf{H} = (H, *, \to, \wedge, \vee, 1)$ be a prelinear semihoop. A map $\delta : H \to H$ is w-admissible iff for all $a, b \in H$:

$$
\begin{aligned}
a \to \delta(a) &= 1, & \delta(\delta(a)) &= a,\\
\delta(a \to b) &\leq a \to \delta(b), & \delta(a * b) &= \delta(\delta(a) * \delta(b)),\\
\delta(a \wedge b) &= \delta(a) \wedge \delta(b), & \delta(a \vee b) &= \delta(a) \vee \delta(b).
\end{aligned}
$$

Observation: the weakened condition allows to get rid of Glivenko equation $\neg\neg(\neg\neg x \to x) = 1$.

Examples

- $\delta_D(a) = a$ for all $a \in H$.
- $\delta_L(a) = 1$ for all $a \in H$.

Generalizing the construction

Let $\mathbf{H} = (H, *, \rightarrow, \wedge, \vee, 1)$ be a prelinear semihoop and $\delta : H \rightarrow H$ be a w-admissible operator.

Let $\mathbf{S}_\delta(\mathbf{H}) = ((\{0\} \times \delta(H)) \cup (\{1\} \times H), \odot, \Rightarrow, \sqcap, \sqcup, (0,1), (1,1))$ where, for all $b, d \in \{0,1\}$ and $(b,x),(d,y) \in (\{0\} \times \delta(H)) \cup (\{1\} \times H)$:

b	d	$\odot$	$\Rightarrow$	$\sqcap$	$\sqcup$
0	0	$(0,1)$	$(1, y \rightarrow x)$	$(0, x \vee y)$	$(0, x \wedge y)$
0	1	$(0, y \rightarrow x)$	$(1,1)$	$(0,x)$	$(1,y)$
1	0	$(0, x \rightarrow y)$	$(0, \delta(x * y))$	$(0,y)$	$(1,x)$
1	1	$(1, x * y)$	$(1, x \rightarrow y)$	$(1, x \wedge y)$	$(1, x \vee y)$

Generalizing the construction

Let $\mathbf{H} = (H, *, \to, \wedge, \vee, 1)$ be a prelinear semihoop and $\delta : H \to H$ be a w-admissible operator.

Let $\mathbf{S}_\delta(\mathbf{H}) = ((\{0\} \times \delta(H)) \cup (\{1\} \times H), \odot, \Rightarrow, \sqcap, \sqcup, (0,1), (1,1))$ where, for all $b, d \in \{0,1\}$ and $(b,x), (d,y) \in (\{0\} \times \delta(H)) \cup (\{1\} \times H)$:

b	d	$\odot$	$\Rightarrow$	$\sqcap$	$\sqcup$
0	0	$(0,1)$	$(1, y \to x)$	$(0, x \vee y)$	$(0, x \wedge y)$
0	1	$(0, y \to x)$	$(1,1)$	$(0,x)$	$(1,y)$
1	0	$(0, x \to y)$	$(0, \delta(x * y))$	$(0,y)$	$(1,x)$
1	1	$(1, x * y)$	$(1, x \to y)$	$(1, x \wedge y)$	$(1, x \vee y)$

$\mathbf{S}_\delta(\mathbf{H})$ is a directly indecomposable SBP_0-algebra and every such algebra is of this form.

Generalizing the construction

Let $\mathbf{H} = (H, *, \to, \wedge, \vee, 1)$ be a prelinear semihoop and $\delta : H \to H$ be a w-admissible operator.

Let $\mathbf{S}_\delta(\mathbf{H}) = ((\{0\} \times \delta(H)) \cup (\{1\} \times H), \odot, \Rightarrow, \sqcap, \sqcup, (0,1), (1,1))$ where, for all $b, d \in \{0,1\}$ and $(b,x), (d,y) \in (\{0\} \times \delta(H)) \cup (\{1\} \times H)$:

b	d	$\odot$	$\Rightarrow$	$\sqcap$	$\sqcup$
0	0	$(0,1)$	$(1, y \to x)$	$(0, x \vee y)$	$(0, x \wedge y)$
0	1	$(0, y \to x)$	$(1,1)$	$(0,x)$	$(1,y)$
1	0	$(0, x \to y)$	$(0, \delta(x * y))$	$(0,y)$	$(1,x)$
1	1	$(1, x * y)$	$(1, x \to y)$	$(1, x \wedge y)$	$(1, x \vee y)$

$\mathbf{S}_\delta(\mathbf{H})$ is a directly indecomposable SBP$_0$-algebra and every such algebra is of this form.

$\mathbf{S}_{\delta_\mathbf{D}}(\mathbf{H})$ is the disconnected rotation of H, $\mathbf{S}_{\delta_\mathbf{L}}(\mathbf{H})$ is the lifting of H.

Generalizing the construction

Let $(\mathbf{B}, \mathbf{H}, \vee_e)$ be a PSH-triple, and let $(b, h) \sim_e (b', h')$ iff

$$b = b', \ \ \neg b \vee_e c = \neg b \vee_e c' \quad \text{and} \quad b \vee_e \delta(c) = b \vee_e \delta(c')$$

$$\mathbf{B} \otimes_e^{\delta} \mathbf{H} = ((B \times H)/\sim_e, \otimes, \Rightarrow, \sqcap, \sqcup, [0,1], [1,1])$$

where, for all $(b, c), (b', c') \in B \times H$:

$(b, c) \odot (b', c') =$
$(b \wedge b', h_{b \vee b'}(1) \wedge h_{b \vee \neg b'}(c' \to c) \wedge h_{\neg b \vee b'}(c \to c') \wedge h_{\neg b \vee \neg b'}(c * c'));$

$(b, c) \Rightarrow (b', c') =$
$(b \to b', h_{b \vee b'} \delta(c' \to c) \wedge h_{b \vee \neg b'}(1) \wedge h_{\neg b \vee b'} \delta(c * c') \wedge h_{\neg b \vee \neg b'}(c \to c'));$

$(b, c) \sqcap (b', c') =$
$(b \wedge b', h_{b \vee b'}(c \vee c') \wedge h_{b \vee \neg b'}(c) \wedge h_{\neg b \vee b'}(c') \wedge h_{\neg b \vee \neg b'}(c \wedge c'));$

$(b, c) \sqcup (b', c') =$
$(b \vee b', h_{b \vee b'}(c \wedge c') \wedge h_{b \vee \neg b'}(c') \wedge h_{\neg b \vee b'}(c) \wedge h_{\neg b \vee \neg b'}(c \vee c')).$

Specifying δ

$\mathbf{B} \otimes_e^{\delta} \mathbf{H}$ is a SBP$_0$-algebra.

For each Boolean algebra $\mathbf{B}$ and prelinear semihoop $\mathbf{H}$:

- $\mathbf{B} \otimes_e^{\delta_L} \mathbf{H}$ is a SMTL-algebra,
- $\mathbf{B} \otimes_e^{\delta_D} \mathbf{H}$ is a IBP$_0$-algebra.

Specifying δ

Let $\mathbb{H}$ be a variety of prelinear semihoops.

Let $\mathbb{T}_{\mathbb{H}}$ be the full subcategory of $\mathbb{T}_{\mathbb{PSH}}$ whose objects are triples $(\mathbf{B}, \mathbf{H}, \vee_e)$ where $\mathbf{H} \in \mathbb{H}$.

Let $\mathbb{SMTL}_{\mathbb{H}}$ and $\mathbb{IBP}_{0\mathbb{H}}$ be the full subcategory respectively of $\mathbb{SMTL}$ and $\mathbb{IBP}_0$ consisting of algebras $\mathbf{A}$ such that the maximum sub-semihoop $\mathbf{H}_{\mathbf{A}} \in \mathbb{H}$.

Suitably generalizing the functors, we can prove the following.

Theorem

Given any $\mathbb{H}$ subvariety of $\mathbb{PSH}$, it holds:

- $\mathbb{SMTL}_{\mathbb{H}}$ *and $\mathcal{T}_{\mathbb{H}}$ are categorically equivalent.*
- $\mathbb{IBP}_{0\mathbb{H}}$ *and $\mathcal{T}_{\mathbb{H}}$ are categorically equivalent.*

Corollary

$\mathbb{SMTL}_{\mathbb{H}}$ *and* $\mathbb{IBP}_{0\mathbb{H}}$ *are categorically equivalent.*

Special cases

PSH-Triples Let $\mathbb{H}$ be the variety of prelinear semihoops.
We have that $\mathbb{SMTL}$ is categorically equivalent to $\mathbb{IBP}_0$.

CH-Triples Let $\mathbb{H}$ be the variety of cancellative hoops.

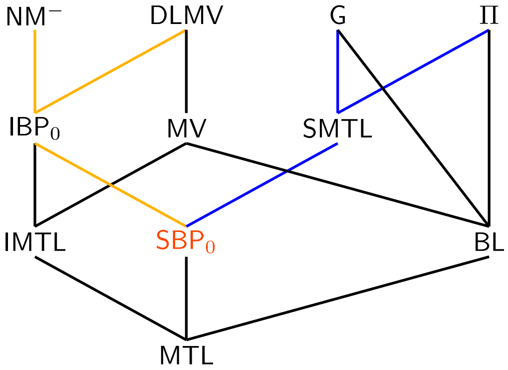
Let $\mathbf{B}$ be a Boolean algebra and $\mathbf{C} \in \mathbb{H}$.

Then $\mathbf{B} \otimes_e^{\delta_L} \mathbf{C}$ is a product algebra, $\mathbf{B} \otimes_e^{\delta_D} \mathbf{C}$ is a DLMV-algebra, and we obtain our first categorical equivalence.

GH-Triples Let $\mathbb{H}$ be the variety of Gödel hoops.

Let $\mathbf{B}$ be a Boolean algebra and $\mathbf{H} \in \mathbb{H}$.

$\mathbf{B} \otimes_e^{\delta_L} \mathbf{H}$ is a Gödel algebra, $\mathbf{B} \otimes_e^{\delta_D} \mathbf{H}$ is a NM^--algebra, and the category of Gödel algebras $\mathbb{G}$ is equivalent to $\mathbb{NM}^-$.


NM$^-$
DLMV
G
Π
IBP$_0$
MV
SMTL
IMTL
SBP$_0$
BL
MTL

Reaching all strongly perfect MTL-algebras

Fix any $\mathbb{H}$ subvariety of $\mathbb{PSH}$, and let $\mathcal{Q}_{\mathbb{H}}$ be the following category:

- The objects are quadruples $(\mathbf{B}, \mathbf{H}, \vee_e, \delta)$ where $\mathbf{H} \in \mathbb{H}$, $(\mathbf{B}, \mathbf{H}, \vee_e) \in \mathcal{T}_{\mathbb{H}}$ and $\delta : H \to H$ w-admissible.
- The morphisms are pairs $(f, g) : (\mathbf{B_1}, \mathbf{H_1}, \vee_e^1, \delta_1) \to (\mathbf{B_2}, \mathbf{H_2}, \vee_e^2, \delta_2)$, such that:
  1. (f, g) is a good morphism pair from $(\mathbf{B_1}, \mathbf{H_1}, \vee_e^1)$ to $(\mathbf{B_2}, \mathbf{H_2}, \vee_e^2)$
  2. for all $x \in H_1$, $g(\delta_1(x)) = \delta_2(g(x))$.

The general equivalence theorem

Let $\mathbb{SBP}_{0\mathbb{H}}$ be the full subcategory of $\mathbb{SBP}_0$ consisting of algebras $\mathbf{A}$ such that $H_A \in \mathbb{H}$.

Again, we can generalize our functors and prove the following.

Theorem

Given any $\mathbb{H}$ subvariety of $\mathbb{PSH}$, $\mathbb{SBP}_{0\mathbb{H}}$ and $\mathcal{Q}_{\mathbb{H}}$ are categorically equivalent.

Hence in particular, $\mathbb{SBP}_0$ and $\mathcal{Q}_{\mathbb{PSH}}$ are categorically equivalent.

Weak Boolean product

Now we focus on the construction.

Given $\mathbf{A}$ a SBP$_0$-algebra, for each $\mathfrak{p} \in \mathrm{Max}\,\mathbf{B_A}$, let

$$\Theta_{\mathfrak{p}} = \{(c, c') \in \mathbf{H_A} \times \mathbf{H_A} \ s.t.\ \exists b \in \mathfrak{p} \mid \neg b \vee_e c = \neg b \vee_e c'\}.$$

Each $\Theta_{\mathfrak{p}}$ is a congruence of $\mathbf{H_A}$, moreover it holds:

Theorem

Every SBP$_0$-algebra $\mathbf{A}$ is a subdirect product of the indexed family

$$\mathbf{B_A}/\mathfrak{p} \otimes_e^{\delta} \mathbf{H_A}/\Theta_{\mathfrak{p}}$$

for some w-admissible operator δ, and for $\mathfrak{p} \in \mathrm{Max}\,\mathbf{B_A}$.

Weak Boolean product

Definition

A weak Boolean product of an indexed family $(\mathbf{A}_x)_{x\in X}$, $X \neq \varnothing$, of algebras is a subdirect product $\mathbf{A} \leq \prod_{x\in X}\mathbf{A_x}$, where X can be endowed with a Boolean space topology such that:

1. $[\![x = y]\!]$ is open for $x, y \in \mathbf{A}$.
2. If $x, y \in \mathbf{A}$ and N is a clopen subset of X, then $x_{|_N} \cup y_{|_{X\setminus N}} \in A$.

If $[\![x = y]\!]$ is clopen, $\mathbf{A} \leq \prod_{x\in X}\mathbf{A_x}$ is a Boolean product.

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If $[\![x = y]\!]$ is clopen, $\mathbf{A} \leq \prod_{x \in X} \mathbf{A_x}$ is a Boolean product.

We can prove that our construction is a weak Boolean product:

Theorem

Every SBP$_0$-algebra $\mathbf{A}$ is a weak Boolean product of the indexed family

$$\mathbf{B_A}/\mathfrak{p} \otimes_e^\delta \mathbf{H_A}/\Theta_\mathfrak{p}$$

for some w-admissible operator δ, and for $\mathfrak{p} \in \operatorname{Max} \mathbf{B_A}$.

Weak Boolean product

In particular we can exhibit the equalizer:

(1) Let $N_b = \{\mathfrak{p} \in \mathrm{Max}\,\mathbf{B_A} \mid b \in \mathfrak{p}\}$ for $b \in \mathbf{B_A}$.
Those sets are the basis of the topology.

We have that the equalizer $[\![x = y]\!]$ is open since it is equal to:

$$O = \left(N_{b_1} \cap N_{b_2} \cap \bigcup_{b \in B_1} N_b\right) \cup \left(N_{\neg b_1} \cap N_{\neg b_2} \cap \bigcup_{\neg b' \in B_2} N_{\neg b'}\right).$$

Where we have $b_1, b_2 \in \mathbf{B_A}$, $c_1, c_2 \in \mathbf{H_A}$ such that:

$x = (\neg b_1 \vee c_1) \wedge (b_1 \vee \neg c_1)$,

$y = (\neg b_2 \vee c_2) \wedge (b_2 \vee \neg c_2)$.

$B_1 = \{b \in \mathbf{B_A} \mid \neg b \vee c_1 = \neg b \vee c_2\}$

$B_2 = \{\neg b' \in \mathbf{B_A} \mid b' \vee \neg c_1 = b' \vee \neg c_2\}$.

Weak Boolean product

Notice that if $\mathbf{B}_{\mathbf{A}}$ is complete,

$$\bigcup_{b \in B_1} N_b = N_{b^*} \text{ and } \bigcup_{\neg b' \in B_2} N_{\neg b'} = N_{b^{**}}$$

where $b^* = \bigvee_{b \in B_1} b$ and $b^{**} = \bigvee_{\neg b' \in B_2} \neg b'$.

Hence, $O = [\![x = y]\!]$ is clopen and we have a Boolean product.

Note: This does not characterize SBP_0-algebras with complete Boolean skeleton. Ex: it also holds in the case that $\mathbf{A}$ is the direct product of the family $\mathbf{B}_{\mathbf{A}}/\mathfrak{p} \otimes_e^{\delta} \mathbf{H}_{\mathbf{A}}/\mathfrak{p}$, for $\mathfrak{p} \in \mathrm{Max}\, \mathbf{B}_{\mathbf{A}}$, where $\mathbf{B}_{\mathbf{A}}$ need not be complete.

Other ideas and future work

- **Generalize the hoop.**

  In order to have our construction, it seems that we only need $\mathbf{H}$ to be a distributive integral lattice.

- **Substitute the Boolean algebra.**

  We need a distributive lattice, whose dual space is compact.
  Finite MV algebras?
  Finite Gödel algebras in order to obtain ordinal sums?