# Structural Completeness in Relevance Logics

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Thinking of a *logic* as a substitution-invariant finitary consequence relation $\vdash$ over sentential formulas in an algebraic signature, we say that $\vdash$ is *structurally complete* if each of its proper extensions has some new theorem (as opposed to having nothing but new rules of derivation). Equivalently, $\vdash$ is structurally complete if it contains all of its *admissible rules*—these are the finite schematic rules under which its set of theorems is closed, cf. [10]. In this case, $\vdash$ has a high degree of self-sufficiency in relation to its meta-theory. (Below, we attribute to a formal system $\mathbf{F}$ the significant properties of its deducibility relation $\vdash_{\mathbf{F}}$.)

The relevance logic $\mathbf{R}$ of Anderson and Belnap is formulated without sentential constants in [1], for the sake of a variable-sharing principle. Its conservative expansion $\mathbf{R}^{\mathbf{t}}$ incorporates the 'Ackermann truth constant' $\mathbf{t}$, and counts among the *substructural* logics of [5]. The adoption of $\mathbf{t}$ is innocuous for most purposes but, as shown below, it makes a considerable difference to the lattice of axiomatic extensions and to questions of structural completeness.

A *De Morgan monoid* $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, \mathbf{t}\rangle$ comprises a distributive lattice $\langle A; \wedge, \vee\rangle$, a commutative monoid $\langle A; \cdot, \mathbf{t}\rangle$ satisfying $x \leqslant x \cdot x$, and a function $\neg\colon A \longrightarrow A$ such that $\boldsymbol{A}$ satisfies $\neg\neg x = x$ and $(x \cdot y \leqslant z \implies \neg z \cdot y \leqslant \neg x)$. It becomes an involutive residuated lattice in the sense of [5] if we define $x \to y = \neg(x \cdot \neg y)$ and $\mathbf{f} = \neg\mathbf{t}$. Every De Morgan monoid satisfies

$$\mathbf{t} \leqslant x \iff x \to x \leqslant x. \tag{1}$$

The $\mathbf{t}$–free subreducts of De Morgan monoids $\boldsymbol{A}$ (i.e., the subalgebras of the reducts $\langle A; \cdot, \wedge, \vee, \neg\rangle$) are called *relevant algebras*. De Morgan monoids and relevant algebras form varieties $\mathsf{DM}$ and $\mathsf{RA}$, respectively.

For each subquasivariety $\mathsf{K}$ of $\mathsf{DM}$, there is a logic $\vdash^{\mathsf{K}}$ with the same signature, defined as follows: for any set $\Gamma \cup \{\alpha\}$ of formulas, $\Gamma \vdash^{\mathsf{K}} \alpha$ iff there exist

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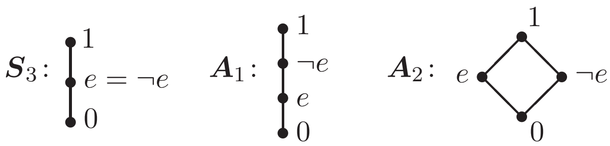
$\gamma_1, \ldots, \gamma_n \in \Gamma$ such that every algebra in $\mathsf{K}$ satisfies

$$\mathbf{t} \leqslant \gamma_1 \wedge \ldots \wedge \gamma_n \implies \mathbf{t} \leqslant \alpha.$$

The map $\mathsf{K} \mapsto \vdash^{\mathsf{K}}$ is a lattice anti-isomorphism from the subquasivarieties of $\mathsf{DM}$ to the extensions of the relevance logic $\mathbf{R}^{\mathbf{t}}$, carrying the subvarieties of $\mathsf{DM}$ onto the axiomatic extensions. (In essence, this was shown in J.M. Dunn's contributions to [1].)

The logic $\mathbf{R}$ lacks the symbol $\mathbf{t}$, but all claims in the previous paragraph remain true when we replace $\mathbf{R}^{\mathbf{t}}$ by $\mathbf{R}$, and $\mathsf{DM}$ by $\mathsf{RA}$, provided we use (1) to eliminate all mention of $\mathbf{t}$.

The variety of Boolean algebras is the smallest nontrivial subquasivariety of $\mathsf{RA}$. This reflects the fact that classical propositional logic (**CPL**) is the largest consistent extension of $\mathbf{R}$. In [11], the second author showed that, in the lattice of axiomatic extensions of $\mathbf{R}$, the unique co-atom **CPL** covers just three logics. The algebraic counterparts of these three logics are the varieties generated, respectively, by the simple relevant algebras $\boldsymbol{S}_3$, $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ depicted below (the first being the 3–element Sugihara algebra).




In each case, $a \cdot 0 = 0$ for all elements $a$, while $e$ is an (undistinguished) neutral element for $\cdot$; its image under $\neg$ is indicated. In $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, we have $\neg e \cdot \neg e = 1$.

For $\mathbf{R}$ (and $\mathbf{R}^{\mathbf{t}}$), structural incompleteness was shown a long time ago in [6]. In general, a logic algebraized by a variety $\mathsf{K}$ is structurally complete iff every proper subquasivariety of $\mathsf{K}$ generates a proper subvariety of $\mathsf{K}$; see [2, 7]. Combining this fact with Świrydowicz' description of the 'almost minimal' subvarieties of $\mathsf{RA}$ and universal algebraic methods, we can prove:

**Theorem 1.** ([9]) *No consistent axiomatic extension of* $\mathbf{R}$ *is structurally complete, except for* **CPL**.

The word 'axiomatic' cannot be omitted in Theorem 1, as the admissible rules of $\mathbf{R}$ axiomatize a structurally complete non-axiomatic extension of $\mathbf{R}$, having the same theorems as $\mathbf{R}$.

In a logic $\vdash$, a rule $\gamma_1, \ldots, \gamma_n/\alpha$ $(n \geq 1)$ is said to be *passive* if no substitution turns all of $\gamma_1, \ldots, \gamma_n$ into theorems of $\vdash$. If $\vdash$ contains all of its passive rules, it is said to be *passively structurally complete*. (This property is studied in [3, 4, 8, 12].

It obtains in all structurally complete systems, because passive rules are vacuously admissible.) The proof of Theorem 1 actually reveals that

$$p,\ p \to (p \to p),\ (\neg p \to p) \to \neg p \ /\ q$$

is an underivable passive rule of each axiomatic consistent extension of **R**, other than **CPL**, so the following strengthening of Theorem 1 holds:

**Theorem 2.** ([9]) *No consistent axiomatic extension of* **R** *is even passively structurally complete, except for* **CPL**.

The situation for $\mathbf{R^t}$ is very different. As will be explained in another talk, $\mathbf{R^t}$ has infinitely many structurally complete axiomatic extensions *incomparable* with **CPL**. Apart from highlighting the impact of **t**, the above findings complement negative results about neighbours of **R** in [7], whose own proofs break down in the context of **R** itself.

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