Structural Completeness in Relevance Logics

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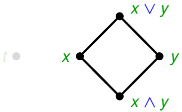
A **De Morgan monoid** $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, t \rangle$ comprises

- a distributive lattice $\langle A; \wedge, \vee \rangle$ (in which $x \leqslant y$ means $x \wedge y = x$);



- a commutative monoid $\langle A; \cdot, t \rangle$ satisfying $x \leqslant x \cdot x$;
- an 'involution' $\neg \colon A \to A$ satisfying $\neg\neg x = x$ and $x \cdot y \leqslant z \implies x \cdot \neg z \leqslant \neg y$ (so $\neg \colon \langle A; \wedge, \vee \rangle \cong \langle A; \vee, \wedge \rangle$).

Defining $x \to y = \neg(x \cdot \neg y)$, we obtain the

Law of Residuation: $x \cdot y \leqslant z \iff x \leqslant y \to z$.

The t-free subreducts of De Morgan monoids $\boldsymbol{A}$ (i.e., the subalgebras of $\langle A; \cdot, \wedge, \vee, \neg \rangle$) are called **relevant algebras**.

$\mathcal{DM} = \{$all De Morgan monoids$\}$ and
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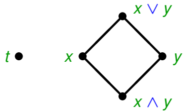

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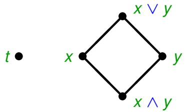

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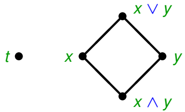

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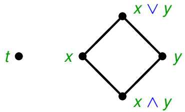

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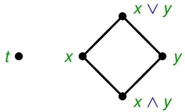

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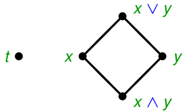

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More generally, there are 'rules': $\gamma_1, \ldots, \gamma_n \vdash_{\mathbf{R^t}} \alpha$ iff

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Example 1. $\vdash_{\mathbf{R^t}} x \to x$, because $\mathcal{DM} \models t \leqslant x \to x$, i.e., $\mathcal{DM} \models t \cdot x \leqslant x$, in fact $\mathcal{DM} \models t \cdot x = x$.

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$\mathbf{R^t}$ and $\mathbf{R}$ have the *same* theorems/rules *not* involving the symbol t, and both are finitely axiomatizable.

They also share many properties with classical logic, *but*

$$\nvdash_{\mathbf{R}} x \to (y \to x) \text{ ('weakening')} \quad \text{and} \quad x \nvdash_{\mathbf{R}} y \to x.$$

Notably, $\mathbf{R}$ (not $\mathbf{R^t}$) has the following 'relevance principle':

if $\vdash_{\mathbf{R}} \alpha \to \beta$, then α and β have a common variable.

Axiomatic extensions of $\mathbf{R^t}$ or of $\mathbf{R}$ correspond in the same way to subvarieties of $\mathcal{DM}$ or of $\mathcal{RA}$, respectively.

In fact, the lattice of axiomatic extensions is isomorphic to the lattice of subvarieties (upside down), so we study the latter, using the tools of universal algebra.

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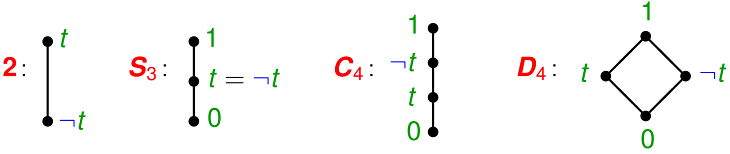

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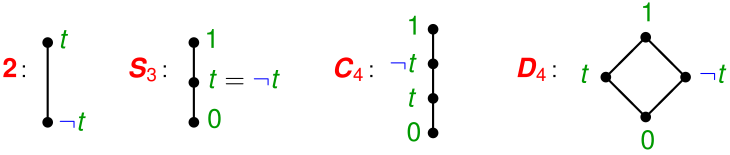

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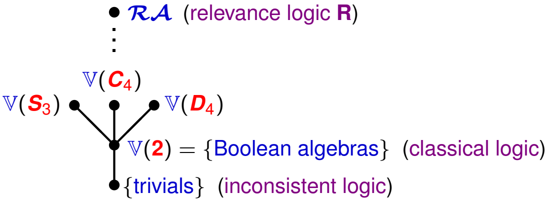

Application:

Let $\mathcal{K} \subseteq \mathcal{RA}$ algebraize an axiomatic extension **L** of **R**.

Defn. A formal rule $\gamma_1, \ldots, \gamma_n \vdash \alpha$ is **admissible** in **L** if its addition to **L** creates no new *theorems* (as opposed to rules).

If every admissible rule of **L** already belongs to **L**, then **L** is said to be **structurally complete**.

E.g., classical logic is structurally complete;

R is not, because Meyer and Dunn (1969) showed:

if $\vdash_{\mathbf{R}} \alpha$ and $\vdash_{\mathbf{R}} (\neg\alpha) \vee \beta$, then $\vdash_{\mathbf{R}} \beta$ ('disjunctive syllogism'),

whereas $x, (\neg x) \vee y \not\vdash_{\mathbf{R}} y$.

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Let $\mathcal{K} \subseteq \mathcal{RA}$ algebraize an axiomatic extension **L** of **R**.

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Theorem. *If a variety $\mathcal{K}$ algebraizes a logic* **L**, *then* **L** *is structurally complete iff every proper subquasivariety of $\mathcal{K}$ generates a proper subvariety of $\mathcal{K}$.* (Cf. Bergman, 1991)

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But consider the existential positive sentence

$$\exists x\,(x \to x = x \quad \& \quad \neg x \to x \leqslant \neg x)$$

This holds in $\boldsymbol{E}$ (take $x = t$) but fails in $\mathbf{2}$.

So it fails in $\boldsymbol{B}$ (as $\mathbf{2}$ is a homomorphic image of $\boldsymbol{B}$).

So, $\boldsymbol{E} \notin \mathbb{Q}(\boldsymbol{B})$, and so $\mathbb{Q}(\boldsymbol{B}) \subsetneq \mathcal{K} = \mathbb{V}(\boldsymbol{B})$, whence

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So, $\boldsymbol{E} \notin \mathbb{Q}(\boldsymbol{B})$, and so $\mathbb{Q}(\boldsymbol{B}) \subsetneq \mathcal{K} = \mathbb{V}(\boldsymbol{B})$, whence

$\mathcal{K}$ is not structurally complete. □

Analyzing the proof, we see that

in every consistent axiomatic extension **L** of **R**, the rule

$$x,\ x \to (x \to x),\ (\neg x \to x) \to \neg x \ \vdash\ y$$

is admissible,

but it does *not* belong to **L** (except when **L** is classical logic).

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