Embedding ℓ-bimonoids into involutive residuated lattices

Adam Přenosil

Institute of Computer Science, Czech Academy of Sciences

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Introduction

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Our goal in this talk will be to generalize this to a substructural setting.

Motivation

BA

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DLat ———— BA
flat

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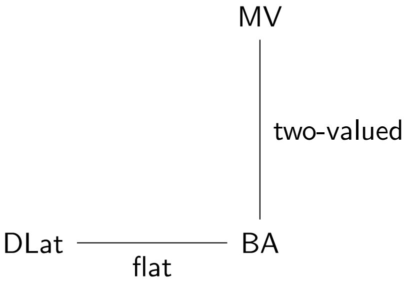

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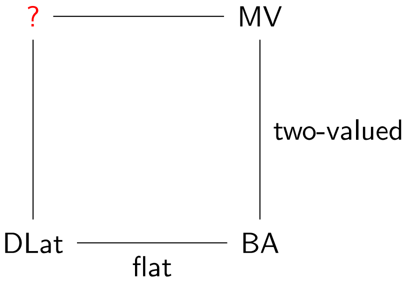

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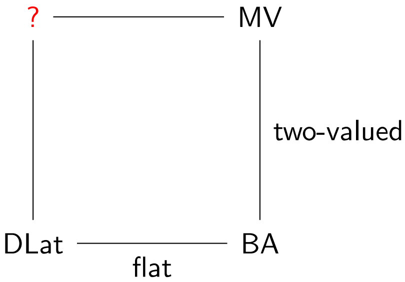

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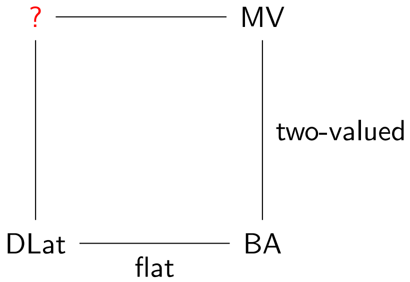

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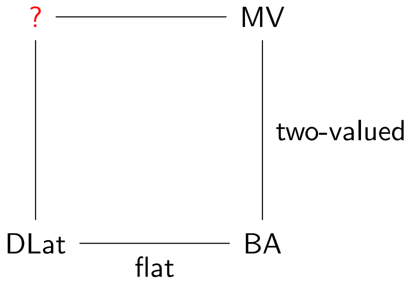

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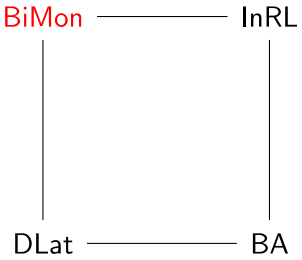
What would many-valued distributive lattices look like?

We do not know. But we can answer a related question. . .

Motivation



Just like distributive lattices are subreducts of Boolean algebras...

...(lattice) ordered bimonoids will be subreducts of involutive RLs.

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We only consider those knotted axioms which hold in **2** (e.g. not $1 \leq x^2$).

Ordered bimonoids

An ordered bimonoid is a pair of ordered monoids $(A, \leq, \cdot, 1)$ and $(A, \geq, +, 0)$ which satisfies the following hemidistributive law:

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The hemidistributive law is an algebraic formulation of cut:

$$\dfrac{\dfrac{\dfrac{a \vdash b, x}{a \leq b + x} \qquad \dfrac{c, x \vdash d}{c \cdot x \leq d}}{a \cdot c \leq (b + x) \cdot c \leq b + (c \cdot x) \leq b + d}}{a, c \vdash b, d}$$

Ordered bimonoids

An ordered bimonoid satisfies a knotted axiom ($knot^n_m$) if:

$$x^n \leq x^m \quad \text{and} \quad mx \leq nx$$

In particular, it is integral if:

$$x \leq 1 \quad \text{and} \quad 0 \leq x$$

and it is contractive if:

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Distributive lattices are precisely the integral contractive bimonoids.

History of the hemidistributive law

The term “hemidistributivity” was introduced by Dunn and Hardegree in their book on *Algebraic Methods in Nonclassical Logics* (2001).

Later studied by Bimbó and Dunn as symmetric gaggles (2009).

Cockett and Seely (1997) use a categorified version of this condition to define weak distributive categories, related to $*$-autonomous categories.

Cockett & Seely and Dunn & Hardegree both point out that it forms an algebraic formulation of the multiple-conclusion cut rule.

The condition was also considered by Grishin (1983) and Moortgat (2007) in the context of the Lambek-style categorial grammar.

Embedding ordered bimonoids into ℓ-bimonoids

Proposition

Each ordered bimonoid embeds into a distributive ℓ-bimonoid.

Proof.

Take the free distributive lattice over the poset and let:

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This lazy construction destroys all existing meets and joins...

...probably a more efficient one is possible.

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Finally, expand this (lattice) ordered monoid by the drastic addition:

$$a + \bot = \bot + a = a \qquad a + b = \top \text{ for } a, b > \bot$$

□

Embedding ℓ-monoids into ℓ-bimonoids

This construction preserves linearity and knotted axioms for multiplication.

Moreover, the addition satisfies all knotted axioms apart from contraction.

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Open problem:
Do all contractive ℓ-monoids embed into contractive ℓ-bimonoids?

Complements in ordered bimonoids

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The unique complement of a, if it exists, will be denoted $-a$.

A bimonoid is complemented if each element has a complement.

Complements in ordered bimonoids

In each complemented ℓ-bimonoid, we may define:

$$a \to b = -a + b$$

This operation is the residual of the multiplication:

$$a \cdot b \leq c \Rightarrow b \leq 1 \cdot b \leq (-a + a) \cdot b \leq -a + (a \cdot b) \leq -a + c$$

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This yields an involutive residuated lattice, i.e. an FL_e-algebra (RL equipped with a constant 0) such that $a = (a \to 0) \to 0$.

Conversely, each involutive RL is a complemented ℓ-bimonoid where:

$$-a = a \to 0$$

Main question

We will be interested in the following question:

Can all ℓ-bimonoids (which satisfy some knotted axioms) be embedded into complemented ℓ-bimonoids (while preserving these knotted axioms)?

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We already know the answer for integral contractive ℓ-bimonoids...

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- embeddability of ℓ-monoids into InRLs

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Terms are build up from constants a and $\overline{a}$ for $a \in \mathbf{A}$, connected by:

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This gives us a term algebra modulo a syntactically defined pre-order.

Factoring out the pre-order yields a complemented ℓ-bimonoid $\mathbf{B}$.

We have $h : \mathbf{A} \to \mathbf{B}$ for free. We need to prove that h is an embedding.

Proof of the main result

The core of the proof is the following:

Lemma

All occurrences of complements may be eliminated from derivations which begin and end with complement-free terms.

Some remarks are in order:

This is *not* cut elimination. Rather, it is akin to identity elimination.

In particular, eliminating complements decreases the length of proofs.

For the sake of simplicity we only illustrate this on ordered bimonoids.

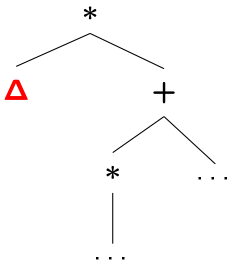
It will be useful to visualize terms as labelled trees.

The rules of the calculus

Monoidal rules: associativity, commutativity, identity

Evaluation rules: $a + b \leq c$ $\quad a \cdot b \leq c$ $\quad a \leq b$

Hemidistribution: $a \cdot (b + c) \leq (a \cdot b) + c$



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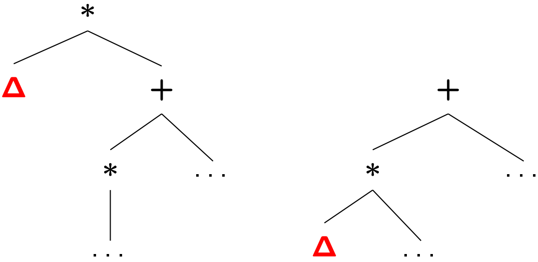

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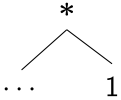
In other words, we may replace the derivation



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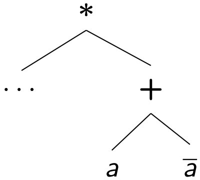

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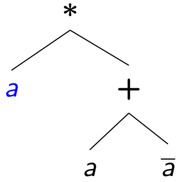

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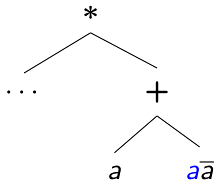

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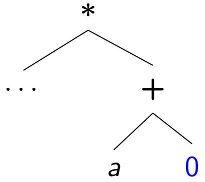

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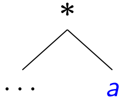

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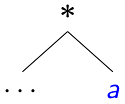

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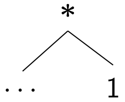

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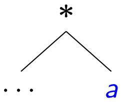

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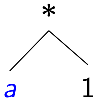

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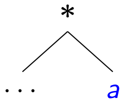

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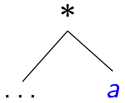


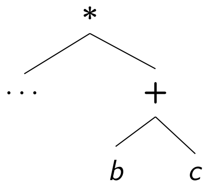
Proof sketch

We need to check that this does not break any instances of knotted rules.

Observation: we never have to pull $a + \overline{a}$ inside $b + \overline{b}$.

Trick: we only use the rule $na \leq ma$, then we get $a^m \leq a^n$ for free.

Example: contractive bimonoid with $ab \leq d$ and $ac \leq d$



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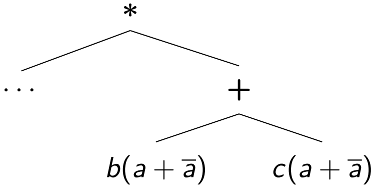

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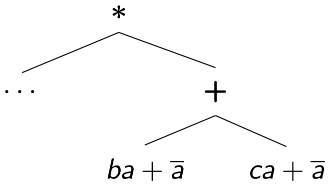

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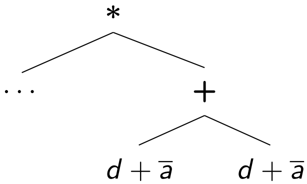

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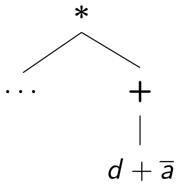

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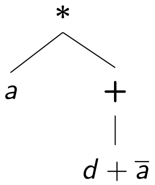

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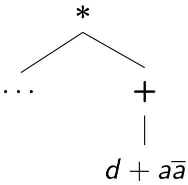

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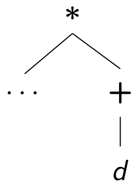

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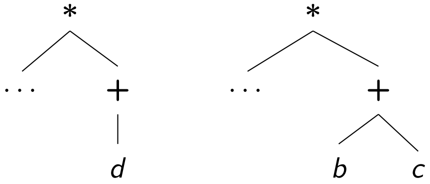

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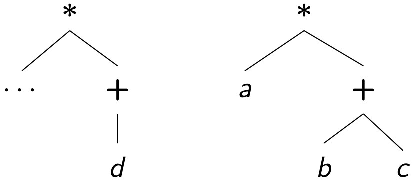

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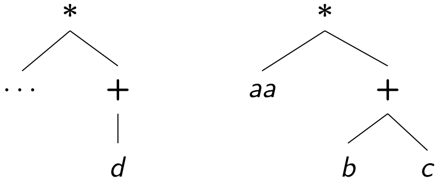

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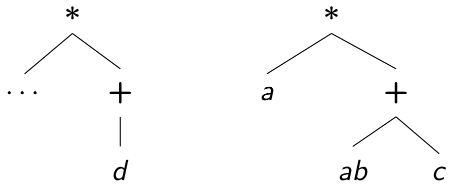

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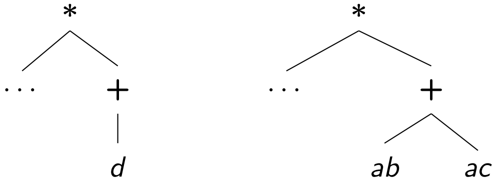

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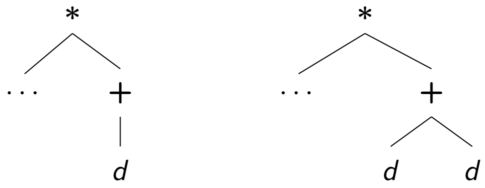

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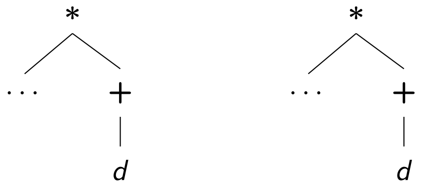

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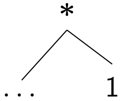

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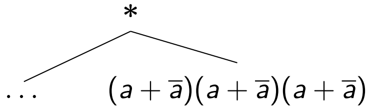

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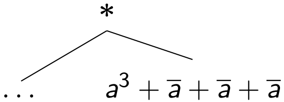

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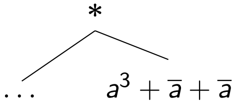

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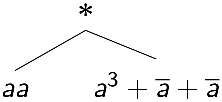

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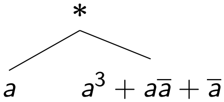

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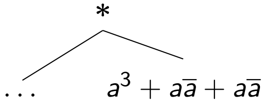

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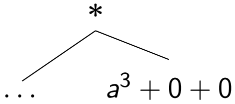

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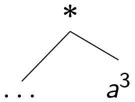

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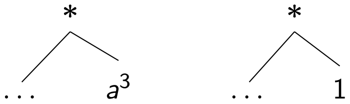

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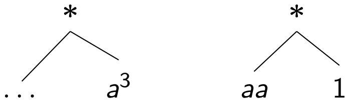

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The proof needs to verify that these transformations go through in any context (if we start with the bottommost instances of complements).

Prelinear InRLs and MV-algebras

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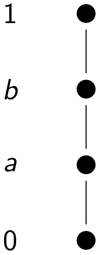
Moreover, MV-algebras are precisely the InRLs which satisfy:

$$a \cdot (b + cd) \leq ab + ac + bd$$

Recall that MV-algebras are the integral prelinear divisible InRLs.

Embeddability into prelinear InRLs

There is a finite s.i. integral linear ℓ-bimonoid which does not embed into any linear (hence into any prelinear) complemented ℓ-bimonoid:



$$a^2 = ab = 0$$

$$b^2 = b$$

$$a + b = b + b = b$$

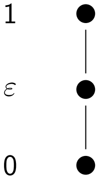
$$a + a = a$$

If this were a subalgebra of a linear complemented ℓ-bimonoid, then either $x \leq -x$ or $-x \leq x$ for all x, hence either $b \cdot b \leq 0$ or $1 \leq b + b$.

Embeddability into MV-algebras

ℓ-bimonoids embeddable into MV-algebras do not form a variety.

Consider the folowing algebra in $\mathbb{HS}(\mathbf{C})$, $\mathbf{C}$ being the Chang algebra:



$$\varepsilon + \varepsilon = \varepsilon$$

$$\varepsilon \cdot \varepsilon = 0$$

Each ℓ-bimonoid embeddable into an MV-algebra satisfies the quasiequation $x \cdot x = x \Rightarrow x + x = x$, unlike the algebra above.

Conclusion

Ordered bimonoids and ℓ-bimonoids are the appropriate structures for the study of complementation in a substructural setting.

The theorem on embedding distributive lattices into Boolean algebras extends to a substructural setting by a syntactical argument.

Moreover, knotted axioms may be preserved by this embedding.

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Thank you for your attention.