# Embedding $\ell$-bimonoids into involutive residuated lattices

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It is a well-known and elementary fact that each distributive lattice embeds into a Boolean algebra. We extend this embedding to a substructural setting by embedding suitable lattice-ordered algebras generalizing distributive lattices into involutive residuated lattices. The crucial observation is that the so-called hemidistributive law introduced by Dunn and Hardegree [2] as an algebraic formulation of the cut rule of sequent calculi provides the appropriate setting for the study of Boolean-like complementation in a substructural context. Since involutive residuated lattices, particularly semilinear ones, form the algebraic semantics of many substructural logics, will yield further insight into implication-free fragments of substructural logics.

An involutive residuated lattice, i.e. a residuated lattice expanded by a point 0 such that $(a \to 0) \to 0 = a$, may be dualized via the operation $-a = a \to 0$ to obtain the monoidal operation $a + b = -(-a \cdot -b)$ and its residual with respect to the dual lattice (see [3]). Exploiting this, we may view involutive residuated lattices as lattice-ordered bimonoids ($\ell$-bimonoids) equipped with a unary complementation operator. We shall be concerned here with embedding $\ell$-bimonoids into complemented $\ell$-bimonoids, i.e. involutive residuated lattices.

We assume all monoids to be commutative throughout. A *partially ordered bimonoid* is a pair of partially ordered monoids $(M_+, \leq, \cdot, 1)$ and $(M_-, \geq, +, 0)$ satisfying:

$$a(b + c) \leq ab + c$$

Here and in the following, we write $ab + c$ for $(a \cdot b) + c$. An *$\ell$-bimonoid* is then a bimonoid ordered by a lattice order such that $a(b \vee c) = ab \vee ac$ and $a + (b \wedge c) = (a + b) \wedge (a + c)$. An *$\ell$-monoid* is a monoid ordered by a lattice order such that $a(b \vee c) = ab \vee ac$.

By *knotted axioms*, we mean the inequalities $a^m \leq a^n$ for $m, n \in \omega$. The inequalities $a \leq 1$ and $a \leq a^2$ are called *integrality* and *contraction*. We say that an $\ell$-bimonoid satisfies an $(m, n)$-knotted axiom if it satisfies both $a^m \leq a^n$ and

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$na \leq ma$. A knotted variety is then a variety axiomatized by a set of knotted axioms (relative to a given base variety). For the sake of simplicity, we exclude the axioms $1 \leq x$ and $x \leq 0$ from consideration in the following. Note that integral contractive $\ell$-bimonoids are precisely distributive lattices.

Let us call $a$ and $b$ *complements* if $ab \leq 0$ and $1 \leq a + b$. The hemidistributive law displayed above ensures that complements, if they exist, are unique. A *complemented $\ell$-bimonoid* shall be an $\ell$-bimonoid in which each element $a$ has a complement, denoted $\overline{a}$. Complemented $\ell$-bimonoids are nothing but ordinary involutive residuated lattices, however, this perspective makes it natural to consider the $\ell$-bimonoidal fragment of the signature of residuated lattice.

Our goal is to describe the $\ell$-bimonoidal fragments of involutive substructural logics axiomatized by knotted axioms or by the prelinearity axiom by showing that each $\ell$-bimonoid can be extended to a complemented $\ell$-bimonoid while preserving the validity of knotted axioms or prelinearity. We use syntactic methods to establish several such embedding theorems.

Before dealing with complements, we first show that it involves no loss of generality to assume the presence of lattice structure. Moreover, the $\ell$-monoidal reducts of knotted varieties of $\ell$-bimonoids generate precisely the corresponding knotted varieties of $\ell$-monoids.

**Theorem 1.** *Each partially ordered bimonoid embeds into a distributive $\ell$-bimonoid. Moreover, this embedding preserves all knotted axioms which hold in the partially ordered bimonoid.*

**Theorem 2.** *Each lattice* $\mathbf{L}$ *embeds into an $\ell$-bimonoid which satisfies all knotted axioms apart from contraction and has a lattice reduct in the quasivariety generated by* $\mathbf{L}$.

**Theorem 3.** *Each $\ell$-monoid embeds into an $\ell$-bimonoid. Moreover, this embedding preserves distributivity as well as all knotted axioms apart from contraction.*

Our main result now states each $\ell$-bimonoid in a given knotted variety of $\ell$-bimonoids embeds into an involutive residuated lattice in the corresponding knotted variety of residuated lattices.

**Theorem 4.** *Each $\ell$-bimonoid embeds into a complemented $\ell$-bimonoid. Moreover, this embedding preserves distributivity and all knotted axioms which hold in the $\ell$-bimonoid.*

Semilinear involutive residuated lattices (InRLs) are another basic class of algebras with a logical significance, being the algebraic semantics of the involutive uninorm logic **IUL**. Recall that an ordered algebra is *semilinear* if it is a subdirect product of totally ordered algebras, and that semilinear residuated lattices are precisely those distributive residuated lattices which satisfy the inequality $1 \leq (a \to b) \vee (b \to a)$. The following observation is now immediate.

**Fact 5.** *Semilinear InRLs are precisely the distributive complemented $\ell$-bimonoids which satisfy the inequality $ab \wedge cd \leq ac \vee bd$, or equivalently $(a+b) \wedge (c+d) \leq (a+c) \vee (b+d)$.*

The last inequality may be viewed as an algebraic formulation of the communication rule in the hypersequent calculi for logics with prelinearity. A similar inequality was used recently by Bou [1] to distinguish the $\ell$-monoidal fragments of the logics **MTL** and **BL**.

We do not have a characterization of $\ell$-bimonoids embeddable into semilinear complemented $\ell$-bimonoids, but the syntactic methods used to obtain the previous embeddings allow us to axiomatize the variety generated by such $\ell$-bimonoids. This axiomatization involves a rather general schema which includes as special cases both of the semilinear inequalities stated above. Due to space restrictions, we therefore merely record the following corollary and content ourselves with showing a simple prototypical example of a non-conservative derivation:

$$(ab+a+b)\wedge 1 \leq (ab+a+b)\wedge(\overline{a}\overline{b}+a+b) \leq (ab\wedge\overline{a}\overline{b})+a+b \leq (a\overline{a}\vee b\overline{b})+a+b \leq 0+a+b \leq a+b$$

**Theorem 6.** *The variety generated by the $\ell$-bimonoidal reducts of (integral) semilinear complemented $\ell$-bimonoids is not finitely axiomatizable.*

Finally, we consider the natural question of whether each semilinear $\ell$-bimonoid embeds into a semilinear complemented $\ell$-bimonoid. Here the answer is negative.

**Fact 7.** *There is a finite subdirectly irreducible totally ordered $\ell$-bimonoid which does not embed into any totally ordered complemented $\ell$-bimonoid.*

# References


[1] Félix Bou. Introducing an exotic MTL-chain. LATD 2014.

[2] J. M. Dunn and G. Hardegree. *Algebraic Methods in Philosophical Logic.* Oxford University Press, New York, 2001.

[3] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics.* Studies in Logic and the Foundations of Mathematics. Elsevier, 2007.