Hyper Natural Deduction[1]

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- Arnon Avron: *Hypersequents, Logical Consequence and Intermediate Logics for Concurrency* Ann.Math.Art.Int. 4 (1991) 225-248

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  - *The name "communication rule" hints, of course, at a certain intuitive interpretation that we have of it as corresponding to the idea of exchanging information between two multiprocesses: [...]*
- General aim: provide Curry-Howard style correspondences for parallel computation, starting from logical systems with good intuitive algebraic / relational semantics.

Setting the stage

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IL $\Leftrightarrow$ ND λ

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Gentzen ’34

SETTING THE STAGE

IL $\Leftrightarrow$ ND λ

introduction rule

$$\frac{A \qquad B}{A \wedge B}$$

elimination rule

$$\frac{A \wedge B}{A}$$

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IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND $\qquad\qquad$ λ

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Sequent	Axiom	Rules
$\Gamma \Rightarrow A$	$A \Rightarrow A$	$\dfrac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \wedge B}$

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(IL) $\Leftrightarrow$ (LJ) $\Leftrightarrow$ (ND) $\qquad$ (λ)

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$$\text{(cut)}\ \frac{\Gamma \Rightarrow A \qquad \Pi, A \Rightarrow B}{\Gamma, \Pi \Rightarrow B}$$

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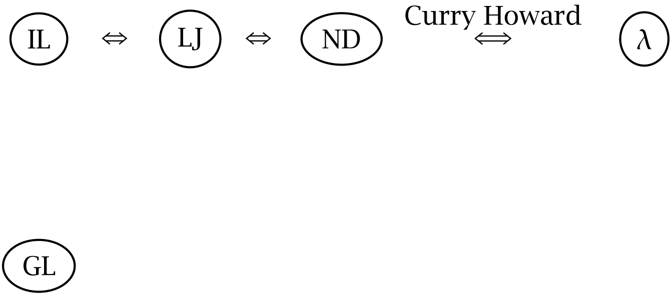
cut elimination – consistency

Setting the stage

$$\text{IL} \Leftrightarrow \text{LJ} \Leftrightarrow \text{ND} \overset{\text{Curry Howard}}{\Longleftrightarrow} \lambda$$

Every proof system hides a model of computation.

Setting the stage



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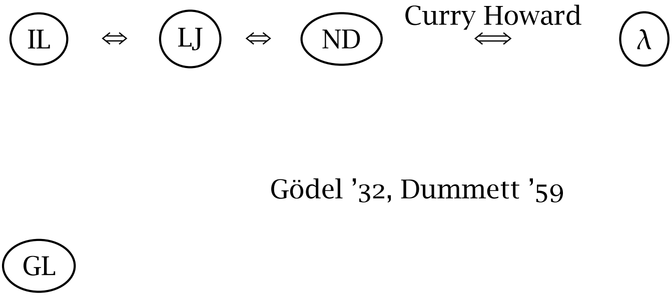

Setting the stage

IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND $\overset{\text{Curry Howard}}{\Longleftrightarrow}$ λ

Gödel '32, Dummett '59

GL

IL + LIN $(A \to B) \vee (B \to A)$

Setting the stage

IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND $\overset{\text{Curry Howard}}{\Longleftrightarrow}$ λ

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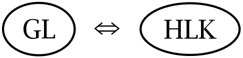
IL + LIN $(A \to B) \vee (B \to A)$

Logic of Linear Kripke Frames

Setting the stage



Avron '91

Setting the stage

IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND $\overset{\text{Curry Howard}}{\Longleftrightarrow}$ λ

GL $\Leftrightarrow$ HLK

Hypersequent

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

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IL $\Leftrightarrow$ LJ $\Leftrightarrow$ ND $\overset{\text{Curry Howard}}{\Longleftrightarrow}$ λ

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$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

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Avron '91: *Communication between agents*

Setting the stage

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Fermüller ’08: Lorenzen style dialogue games, . . .

Setting the stage

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$$\text{(com)}\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

hyper sequent calculi for various logics

Setting the stage

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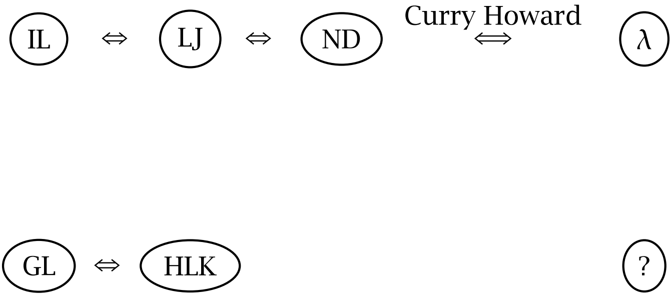
GL $\Leftrightarrow$ HLK

$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

deep inference

Setting the stage



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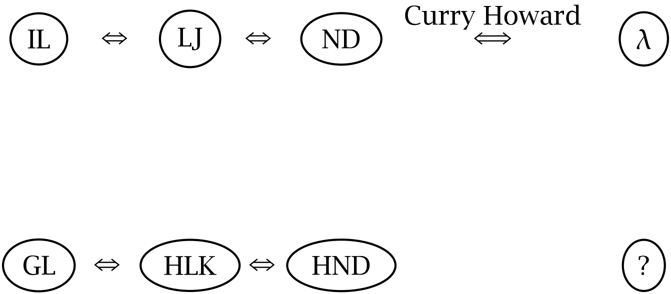

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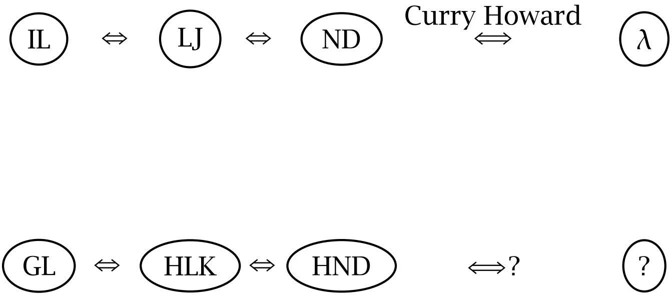

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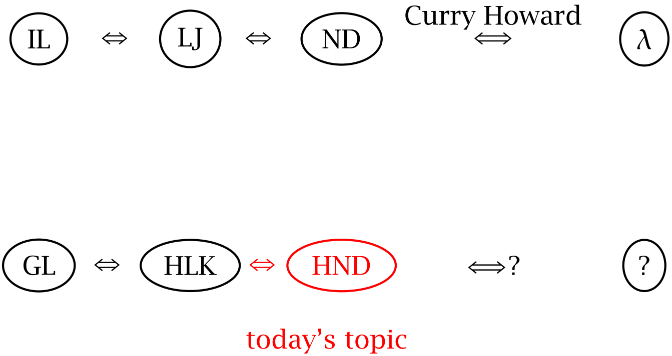

Previous work

Hirai, FLOPS 2012

A Lambda Calculus for Gödel-Dummett Logics Capturing Waitfreedom

- change of both syntax and semantics
- different calculus

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Baaz, Ciabattoni, Fermüller 2000

A Natural Deduction System for Intuitionistic Fuzzy Logic
(will be discussed later)

WISHLIST

Properties we want to have:

(semi) local

- construction of deductions:
  apply ND inspired rules to extend a HND deductions
- modularity of deductions:
  reorder/restructure deductions
- analyticity (sub-formula property, ...)

Wishlist

Properties we want to have:

(semi) local

- construction of deductions:
  apply ND inspired rules to extend a HND deductions
- modularity of deductions:
  reorder/restructure deductions
- analyticity (sub-formula property, . . .)

normalisation

- procedural normalisation via conversion steps

Natural Deduction rules

$$\wedge\text{-}i\ \frac{A \qquad B}{A \wedge B} \qquad\qquad \wedge\text{-}e\ \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$$\vee\text{-}i\ \frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad \vee\text{-}e\ \frac{A \vee B \qquad \begin{matrix}[A]\\ C\end{matrix} \qquad \begin{matrix}[B]\\ C\end{matrix}}{C}$$

$$\rightarrow\text{-}i\ \frac{\begin{matrix}[A]\\ B\end{matrix}}{A \rightarrow B} \qquad\qquad \rightarrow\text{-}e\ \frac{A \qquad A \rightarrow B}{B}$$

$$\bot_I\ \frac{\bot}{A}$$

Hypersequent Calculus

Hypersequent: $\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_n \Rightarrow A_n$

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Some rules:

$$\to,l\ \frac{\Gamma \Rightarrow A \mid \mathcal{H} \qquad \Gamma, B \Rightarrow C \mid \mathcal{H}'}{\Gamma, A \to B \Rightarrow C \mid \mathcal{H} \mid \mathcal{H}'}$$

$$\to,r\ \frac{\Gamma, A \Rightarrow B \mid \mathcal{H}}{\Gamma \Rightarrow A \to B \mid \mathcal{H}}$$

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$$\to,r\ \frac{\Gamma, A \Rightarrow B \mid \mathcal{H}}{\Gamma \Rightarrow A \to B \mid \mathcal{H}}$$

$$com\ \frac{\Gamma_1 \Rightarrow A_1 \mid \mathcal{H} \qquad \Gamma_2 \Rightarrow A_2 \mid \mathcal{H}'}{\Gamma_1 \Rightarrow A_2 \mid \Gamma_2 \Rightarrow A_1 \mid \mathcal{H} \mid \mathcal{H}'}$$

$$split\ \frac{\Pi, \Gamma \Rightarrow A \mid \mathcal{H}}{\Pi \Rightarrow A \mid \Gamma \Rightarrow A \mid \mathcal{H}}$$

Linearity in LJ

$$\frac{}{\Rightarrow (A \to B) \vee (B \to A)}$$

Linearity in LJ

$$\dfrac{\overline{\Rightarrow A \rightarrow B}}{\Rightarrow (A \rightarrow B) \vee (B \rightarrow A)}$$

Linearity in LJ

$$\dfrac{\dfrac{\overset{???}{A \Rightarrow B}}{\Rightarrow A \to B}}{\Rightarrow (A \to B) \vee (B \to A)}$$

Linearity in HLK

$$\frac{}{\Rightarrow (A \to B) \vee (B \to A)}$$

Linearity in HLK

$$\dfrac{\overline{\Rightarrow (A \to B) \vee (B \to A) \mid \ \Rightarrow (A \to B) \vee (B \to A)}}{\Rightarrow (A \to B) \vee (B \to A)}\ \text{EC}$$

Linearity in HLK

$$\dfrac{\dfrac{}{\Rightarrow (A \to B) \vee (B \to A) \mid\ \Rightarrow B \to A}\ \vee\text{-r}}{\dfrac{\Rightarrow (A \to B) \vee (B \to A) \mid\ \Rightarrow (A \to B) \vee (B \to A)}{\Rightarrow (A \to B) \vee (B \to A)}\ \text{EC}}$$

LINEARITY IN HLK

$$
\dfrac{\dfrac{\dfrac{\overline{\Rightarrow A \to B \mid\ \Rightarrow B \to A}}{\Rightarrow (A \to B) \vee (B \to A) \mid\ \Rightarrow B \to A}\ \vee\text{-r}}{\Rightarrow (A \to B) \vee (B \to A) \mid\ \Rightarrow (A \to B) \vee (B \to A)}\ \vee\text{-r}}{\Rightarrow (A \to B) \vee (B \to A)}\ \text{EC}
$$

Linearity in HLK

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Rightarrow A \to B \mid B \Rightarrow A}}{\Rightarrow A \to B \mid\ \Rightarrow B \to A}\ \to\text{-r}}{\Rightarrow (A \to B) \vee (B \to A) \mid\ \Rightarrow B \to A}\ \vee\text{-r}}{\Rightarrow (A \to B) \vee (B \to A) \mid\ \Rightarrow (A \to B) \vee (B \to A)}\ \vee\text{-r}
$$

$$
\dfrac{\Rightarrow (A \to B) \vee (B \to A) \mid\ \Rightarrow (A \to B) \vee (B \to A)}{\Rightarrow (A \to B) \vee (B \to A)}\ \text{EC}
$$

Linearity in HLK

$$
\begin{array}{cl}
\overline{A \Rightarrow B \mid B \Rightarrow A} & \\
\hline
\Rightarrow A \to B \mid B \Rightarrow A & \to\text{-r} \\
\hline
\color{red}{\Rightarrow A \to B \mid \; \Rightarrow B \to A} & \to\text{-r} \\
\hline
\Rightarrow (A \to B) \vee (B \to A) \mid \; \Rightarrow B \to A & \vee\text{-r} \\
\hline
\Rightarrow (A \to B) \vee (B \to A) \mid \; \Rightarrow (A \to B) \vee (B \to A) & \vee\text{-r} \\
\hline
\color{red}{\Rightarrow (A \to B) \vee (B \to A)} & \text{EC}
\end{array}
$$

Linearity in HLK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}\ com}{\Rightarrow A \to B \mid B \Rightarrow A}\ \to\text{-r}}{\Rightarrow A \to B \mid\ \Rightarrow B \to A}\ \to\text{-r}}{\Rightarrow (A \to B) \vee (B \to A) \mid\ \Rightarrow B \to A}\ \vee\text{-r}}{\Rightarrow (A \to B) \vee (B \to A) \mid\ \Rightarrow (A \to B) \vee (B \to A)}\ \vee\text{-r}}{\Rightarrow (A \to B) \vee (B \to A)}\ \text{EC}
$$

Various different systems

BCF System

Models hyper sequents in natural deduction by combining deductions in ND with a new operator $|$.

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Example linearity: From

$$\text{From} \quad \frac{\{A\}}{A} \quad \text{and} \quad \frac{\{B\}}{B}$$

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Example linearity: From

$$\text{From} \quad \frac{\{A\}}{A} \quad \text{and} \quad \frac{\{B\}}{B}$$

$$\text{one derives} \quad (com)\frac{\dfrac{\{A\}}{A}}{B} \quad \Bigg| \quad (com)\frac{\dfrac{\{B\}}{B}}{A}$$

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Example linearity: From

$$\text{From} \quad \frac{\{A\}}{A} \quad \text{and} \quad \frac{\{B\}}{B}$$

$$\text{one derives} \quad (com)\,\frac{\dfrac{\{A\}}{A}}{B} \;\Bigg|\; (com)\,\frac{\dfrac{\{B\}}{B}}{A}$$

$$\text{then} \quad \frac{(com)\,\dfrac{\dfrac{\{A\}}{A}}{B}}{A \to B} \;\Bigg|\; \frac{(com)\,\dfrac{\dfrac{\{B\}}{B}}{A}}{B \to A} \quad \text{etc}$$

Discussion of the BCF system

- direct translation from HLK
- inductive definition
- easy to translate proofs back and forth
- normalisation only via translation to HLK

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Not a solution to our problem!

Our approach to Hyper Natural Deduction

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$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

$$\begin{array}{c}\Gamma \\ \vdots \\ A\end{array}$$

Our approach to Hyper Natural Deduction

$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad\qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \Delta \Rightarrow A}$$

$$\begin{array}{ccc} \Gamma & \qquad\qquad & \Delta \\ \vdots & & \vdots \\ A & & B \end{array}$$

Our approach to Hyper Natural Deduction

$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\color{red}{\Gamma \Rightarrow B} \mid \Delta \Rightarrow A}$$

$$\begin{array}{cc} \Gamma & \Delta \\ \vdots & \vdots \\ \color{red}{com}\ \dfrac{A}{\color{red}{B}} & B \end{array}$$

OUR APPROACH TO HYPER NATURAL DEDUCTION

$$(\text{com})\ \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \Rightarrow B \mid \color{red}{\Delta \Rightarrow A}}$$

$$\begin{array}{cc} \Gamma & \Delta \\ \vdots & \vdots \\ com\ \dfrac{A}{B} & \color{red}{\overline{com}}\ \dfrac{B}{\color{red}{A}} \end{array}$$

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Our approach to Hyper Natural Deduction

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$$\begin{array}{c}\Gamma \\ \vdots \\ com\ \dfrac{A}{B}\end{array} \quad \begin{array}{c}\Delta \\ \vdots \\ \overline{com}\ \dfrac{B}{A}\end{array}$$

- consider sets of derivation trees
- divide communication (and split) into two dual parts
- search for minimal set of conditions that provides sound and complete deduction system

Reasoning in Hyper Natural Deduction

Double extension in *the spirit* of ND:

- from one tree to set of trees
- additional rules

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$$\text{From} \quad \begin{array}{c} \vdots \\ A \end{array} \quad \text{and} \quad \begin{array}{c} \vdots \\ B \end{array} \quad \text{form} \quad \text{com}^{x}_{A,B}\ \dfrac{\begin{array}{c} \vdots \\ A \end{array}}{B} \quad \text{com}^{\bar{x}}_{B,A}\ \dfrac{\begin{array}{c} \vdots \\ B \end{array}}{A}$$

Reasoning in Hyper Natural Deduction

Double extension in *the spirit* of ND:

- from one tree to set of trees
- additional rules

From $\begin{array}{c}\vdots\\ A\end{array}$ and $\begin{array}{c}\vdots\\ B\end{array}$ form $\mathrm{com}^{x}_{A,B}\ \dfrac{\begin{array}{c}\vdots\\ A\end{array}}{B}$ $\quad \mathrm{com}^{\bar{x}}_{B,A}\ \dfrac{\begin{array}{c}\vdots\\ B\end{array}}{A}$

From $\begin{array}{c}\Gamma,\Delta\\ \sigma\,\vdots\\ A\end{array}$ form $x\colon \mathrm{Spt}_{\Gamma,\Delta}\ \dfrac{\begin{array}{c}[\Gamma],\Delta\\ \sigma\,\vdots\\ A\end{array}}{A}$ $\quad \bar{x}\colon \mathrm{Spt}_{\Delta,\Gamma}\ \dfrac{\begin{array}{c}\Gamma,[\Delta]\\ \sigma\,\vdots\\ A\end{array}}{A}$

Beauty of this system

- Hyper rules – derivations are completely in ND style
- Hyper rules mimic HLK/BCF system
- natural style of deduction

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We need criteria to check whether a set of trees forms a proof!

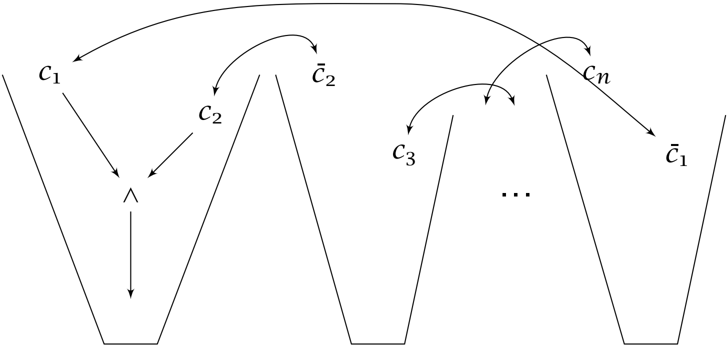
Proof criteria

What about the following proof part:

$$
\dfrac{\dfrac{\dfrac{\vdots}{B}}{A}\,\mathrm{com}^{x}_{B,A} \qquad \dfrac{\dfrac{\vdots}{A}}{B}\,\mathrm{com}^{\bar{x}}_{A,B}}{\dfrac{\vdots}{E} \qquad\qquad \dfrac{\vdots}{F}}
$$

$$
\dfrac{E \qquad\qquad F}{E \wedge F}
$$

Equivalence classes



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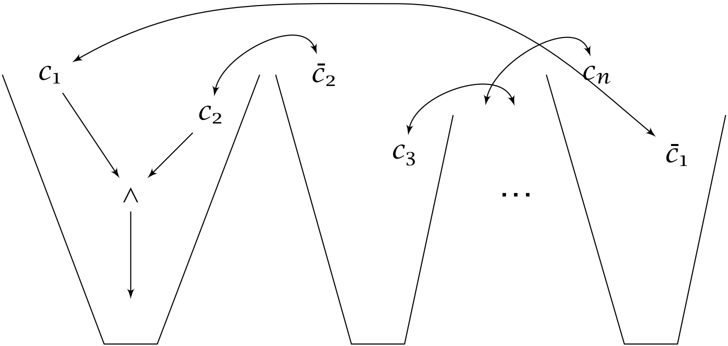

Criterion 1: The sets of trees connected to the sub-trees routed in the predecessors of any non-unary logical rule need to be disjoint.

Another criteria

What about this:

$$\begin{array}{cc} \dfrac{\begin{array}{c}\vdots\\ B\end{array}}{\begin{array}{c}A\\ \vdots\end{array}}\ \mathrm{com}^{x}_{B,A} & \dfrac{\begin{array}{c}\vdots\\ F\end{array}}{\begin{array}{c}E\\ \vdots\end{array}}\ \mathrm{com}^{\bar{x}}_{F,E} \\ \dfrac{E}{\begin{array}{c}F\\ \vdots\end{array}}\ \mathrm{com}^{x}_{E,F} & \dfrac{A}{\begin{array}{c}B\\ \vdots\end{array}}\ \mathrm{com}^{\bar{x}}_{A,B} \end{array}$$

Another criteria

What about this:

$$\begin{array}{cc} \vdots & \vdots \\ \mathrm{com}^{x}_{B,A}\ \dfrac{B}{A} & \mathrm{com}^{\bar{x}}_{F,E}\ \dfrac{F}{E} \\ \vdots & \vdots \\ \mathrm{com}^{x}_{E,F}\ \dfrac{E}{F} & \mathrm{com}^{\bar{x}}_{A,B}\ \dfrac{A}{B} \\ \vdots & \vdots \end{array}$$

Criterion 2: There is a total order on communication and split labels that is compatible with the order on the branches.

Properties

With a few house-keeping conditions (well-formedness, consistency of naming etc) we obtain a nice natural deduction system with the following properties:

- Hyper rules – derivations are completely in ND style
- Hyper rules mimic HLK/BCF system
- natural style of deduction
- implicit (procedural) definition and explicit (properties on figures) definitions

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PROPERTIES

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- implicit (procedural) definition and explicit (properties on figures) definitions
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- … but what about normalisation

Normalisation

Idea: Reorder deductions where an introduction rule is followed by an elimination rule:

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$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}\;{\to}\text{-i} \qquad \begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}}{B}\;{\to}\text{-e}$$

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Idea: Reorder deductions where an introduction rule is followed by an elimination rule:

$$
{\to}\text{-e}\ \dfrac{{\to}\text{-i}\ \dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B} \qquad \begin{matrix}\Gamma\\ \vdots\\ A\end{matrix}}{B}
\qquad \text{converts to} \qquad
\begin{matrix}\Gamma\\ \vdots\\ A\\ \vdots\\ B\end{matrix}
$$

Normalisation

Idea: Reorder deductions where an introduction rule is followed by an elimination rule:

$$
\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \to B}\,\to\text{-i} \qquad \begin{array}{c}\Gamma\\ \vdots \\ A\end{array}}{B}\,\to\text{-e}
\qquad \text{converts to} \qquad
\begin{array}{c}\Gamma\\ \vdots \\ A \\ \vdots \\ B\end{array}
$$

Effect of normalisation: hourglass form of derivation, eliminations followed by introductions.

Permutation Conversions for GLHN

Example conversion for normalisation in GLHN:

$$
\mathrm{com}^{x}_{A\to B,C}\ \dfrac{\begin{array}{c}\Gamma\\ \sigma_0\,\vdots\\ A\to B\end{array}}{C}
$$

$$
\to\text{-}e\ \dfrac{\mathrm{com}^{\bar{x}}_{C,A\to B}\ \dfrac{\begin{array}{c}\Delta\\ \sigma_1\,\vdots\\ C\end{array}}{A\to B} \qquad \begin{array}{c}\Pi\\ \sigma_2\,\vdots\\ A\end{array}}{B}
$$

Permutation Conversions for GLHN

Example conversion for normalisation in GLHN:

$$
\text{com}^{x}_{A\to B,C}\ \frac{\begin{array}{c}\Gamma\\ \sigma_0\ \vdots\\ A \to B\end{array}}{C}
\qquad
\to\text{-}e\ \frac{\text{com}^{\bar{x}}_{C,A\to B}\ \dfrac{\begin{array}{c}\Delta\\ \sigma_1\ \vdots\\ C\end{array}}{A \to B} \qquad \begin{array}{c}\Pi\\ \sigma_2\ \vdots\\ A\end{array}}{B}
$$

first try: use dual labels as channels to communicate sub-derivations

$$
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Permutation Conversions for GLHN

Example conversion for normalisation in GLHN:

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converts to (similar to cut-elimination in HLK)

$$\mathrm{com}^{x}_{B,C}\ \dfrac{{}^{1}S^{y}_{\Gamma,\Pi}\ \dfrac{\rightarrow\text{-}e\ \dfrac{\begin{array}{c}\Gamma\\ \sigma_0\,\vdots\\ A\to B\end{array} \qquad \begin{array}{c}{}^{1}[\Pi]\\ \sigma_2\,\vdots\\ A\end{array}}{B}}{B}}{C} \qquad contr\ \dfrac{{}^{2}S^{\bar{y}}_{\Pi,\Gamma}\ \dfrac{\rightarrow\text{-}e\ \dfrac{\begin{array}{c}{}^{2}[\Gamma]\\ \sigma_0\,\vdots\\ A\to B\end{array} \qquad \begin{array}{c}\Pi\\ \sigma_2\,\vdots\\ A\end{array}}{B}}{B} \qquad \mathrm{com}^{\bar{x}}_{C,B}\ \dfrac{\begin{array}{c}\Delta\\ \sigma_1\,\vdots\\ C\end{array}}{B}}{B}$$

Those nasty contractions

Problems with contractions

Consider the following proof containing two derivation trees:

$$\dfrac{4{:}\,\mathrm{Ctr}\;\dfrac{\mathrm{com}^{1}_{D,A}\,\dfrac{D}{A}\qquad \mathrm{com}^{2}_{C,A}\,\dfrac{C}{A}}{A}\qquad \mathrm{com}^{3}_{C,B}\,\dfrac{C}{B}}{A \wedge B}$$

$$\dfrac{6{:}\,\mathrm{Ctr}\;\dfrac{\mathrm{com}^{\bar{3}}_{B,C}\,\dfrac{B}{C}\qquad \mathrm{com}^{\bar{2}}_{A,C}\,\dfrac{A}{C}}{C}\qquad \mathrm{com}^{\bar{1}}_{A,D}\,\dfrac{A}{D}}{C \wedge D}$$

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$$\dfrac{6{:}\,\mathrm{Ctr}\ \dfrac{\mathrm{com}^{\bar{3}}_{B,C}\ \dfrac{B}{C} \qquad \mathrm{com}^{\bar{2}}_{A,C}\ \dfrac{A}{C}}{C} \qquad \mathrm{com}^{\bar{1}}_{A,D}\ \dfrac{A}{D}}{C \wedge D}$$

Difficult to read ...

Graph theoretic view

Define an associated graph:

- rules are nodes occurrences, formulas are edges
- labels are label occurrences
- bi-directional edges between dually labeled vertices
- mark up only name of rules

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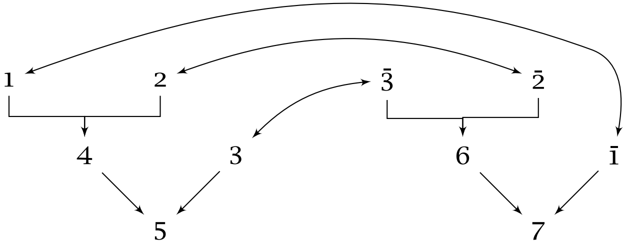

What is the problem?

Graph theoretic view

Define an associated graph:

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- labels are label occurrences
- bi-directional edges between dually labeled vertices
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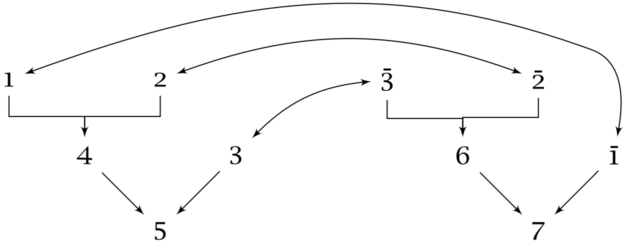

Criterion 1 does not hold in the above graph

Rethinking our initial system

Problems

- considering equivalence classes of trees is not enough
- contractions need to be handled separately (e.g., cutting)

Rethinking our initial system

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Solutions

- switch to graph representation
- accessibility relation in the associated graph
- cut operation and special conditions on contraction ordering

Hyper Natural Deduction

Canopy graphs

Two operations on labeled directed graphs:

$\text{Cut}(\mathcal{G}, E)$	drops a set of edges from the graph
$\text{Drop}(\mathcal{G}, N)$	drops a set of nodes and related edges that are reachable from all nodes labeled with a name in N

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Definition

Let $\mathcal{G} = (V, E, N, f)$ be a labeled graph, and let $E^c \subseteq E$ be the set of symmetric edges, that is the set of all edges $(r, s) \in E$ where also $(s, r) \in E$. If $\mathrm{Cut}(\mathcal{G}, E^c)$ is a disjoint union of trees, we call $\mathcal{G}$ a *C-graph* or *canopy graph*.

Motivation of these concepts

Consider the following hyper-sequent derivation:

$$
\dfrac{
\textit{com}_1\,\dfrac{\dfrac{B \Rightarrow B}{C, B \Rightarrow B} \qquad A \Rightarrow A}{C, B \Rightarrow A \mid A \Rightarrow B}
\qquad
\textit{com}_2\,\dfrac{\dfrac{C \Rightarrow C}{C, B \Rightarrow C} \qquad A \Rightarrow A}{C, B \Rightarrow A \mid A \Rightarrow C}
}{
\textit{contr}\,\dfrac{C, B \Rightarrow A \mid C, B \Rightarrow A \mid A \Rightarrow B \wedge C}{\dfrac{C, B \Rightarrow A \mid A \Rightarrow B \wedge C}{\Rightarrow C \rightarrow (B \rightarrow A) \mid \Rightarrow A \rightarrow B \wedge C}}
}\ \wedge\text{-}r
$$

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$$
\dfrac{contr\;\dfrac{\wedge\text{-}r\;\dfrac{com_1\;\dfrac{\dfrac{B \Rightarrow B}{C, B \Rightarrow B} \qquad A \Rightarrow A}{C, B \Rightarrow A \mid A \Rightarrow B} \qquad com_2\;\dfrac{\dfrac{C \Rightarrow C}{C, B \Rightarrow C} \qquad A \Rightarrow A}{C, B \Rightarrow A \mid A \Rightarrow C}}{C, B \Rightarrow A \mid C, B \Rightarrow A \mid A \Rightarrow B \wedge C}}{C, B \Rightarrow A \mid A \Rightarrow B \wedge C}}{\Rightarrow C \to (B \to A) \mid \; \Rightarrow A \to B \wedge C}
$$

And the following intended HND proof:

$$
w\;\dfrac{z\;\dfrac{y{:}\mathrm{Ctr}\;\dfrac{\mathrm{com}^{x_1}_{C,A}\;\dfrac{[C]}{A} \qquad \mathrm{com}^{x_2}_{B,A}\;\dfrac{[B]}{A}}{A}}{B \to A}}{C \to (B \to A)}
\qquad\qquad
v\;\dfrac{u{:}\wedge\text{-}i\;\dfrac{\mathrm{com}^{\bar{x}_1}_{A,C}\;\dfrac{[A]}{C} \qquad \mathrm{com}^{\bar{x}_2}_{A,B}\;\dfrac{[A]}{B}}{B \wedge C}}{A \to (B \wedge C)}
$$

Motivation of these concepts II

$$
w\;\dfrac{z\;\dfrac{y{:}\,\mathrm{Ctr}\;\dfrac{\mathrm{com}^{x_1}_{C,A}\;\dfrac{[C]}{A}\qquad \mathrm{com}^{x_2}_{B,A}\;\dfrac{[B]}{A}}{A}}{B \to A}}{C \to (B \to A)}
$$

$$
v\;\dfrac{u{:}\wedge\text{-}i\;\dfrac{\mathrm{com}^{\bar{x}_1}_{A,C}\;\dfrac{[A]}{C}\qquad \mathrm{com}^{\bar{x}_2}_{A,B}\;\dfrac{[A]}{B}}{B \wedge C}}{A \to (B \wedge C)}
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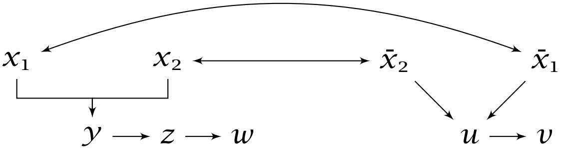
$$

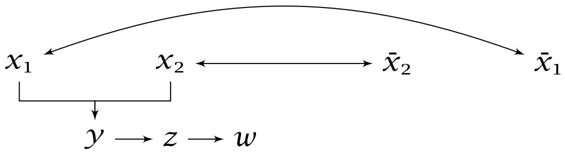
and the associated graph



Motivation of these concepts II

$$ w\;\dfrac{z\;\dfrac{y{:}\,\mathrm{Ctr}\;\dfrac{\mathrm{com}^{x_1}_{C,A}\,\dfrac{[C]}{A}\qquad \mathrm{com}^{x_2}_{B,A}\,\dfrac{[B]}{A}}{A}}{B \to A}}{C \to (B \to A)} \qquad\qquad v\;\dfrac{u{:}\wedge\text{-}i\;\dfrac{\mathrm{com}^{\bar{x}_1}_{A,C}\,\dfrac{[A]}{C}\qquad \mathrm{com}^{\bar{x}_2}_{A,B}\,\dfrac{[A]}{B}}{B \wedge C}}{A \to (B \wedge C)} $$

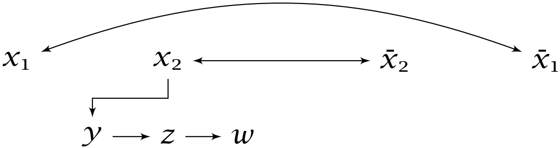
connectivity condition does not hold for u



Motivation of these concepts II

$$w\ \cfrac{z\ \cfrac{y{:}\,\mathrm{Ctr}\ \cfrac{\mathrm{com}^{x_1}_{C,A}\ \dfrac{[C]}{A} \qquad \mathrm{com}^{x_2}_{B,A}\ \dfrac{[B]}{A}}{A}}{B \to A}}{C \to (B \to A)} \qquad\qquad v\ \cfrac{u{:}\wedge\text{-}i\ \cfrac{\mathrm{com}^{\bar{x}_1}_{A,C}\ \dfrac{[A]}{C} \qquad \mathrm{com}^{\bar{x}_2}_{A,B}\ \dfrac{[A]}{B}}{B \wedge C}}{A \to (B \wedge C)}$$

cut at the contraction, conn. comp. fall apart



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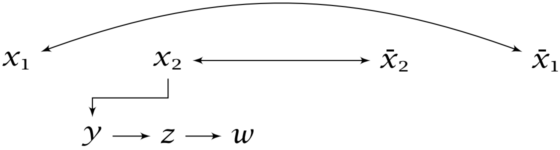

Expresses an implicit ordering between the conjunction (introduced first) and the contraction (introduced later).

Definition of HND for Gödel logics (GLHN)

A finite set of pre-derivations R (together with a total order on labels) forms a GLHN iff

- some obvious consistency conditions are satisfied;
  like occurrence of dual labels, compatibility with fixed label order, …

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- Local dependence of contraction premises r: The connected components in $\mathrm{Cut}(\mathrm{Drop}(\mathcal{G}(R), r))$ of premises of r are equal.

Key ideas of GLHN

- dual communication and splitting labels connect different pre-derivations (trees)
- one pre-derivation (tree) derives one sequent of an hyper-sequent
- connected components of a GLHN (set of trees) prove one hyper-sequent
- conditions on the independence of connected components for non-unary rule occurrences

Linearity in GLHN

$$\frac{}{(A \to B) \vee (B \to A)}$$

LINEARITY IN GLHN

$$\dfrac{\dfrac{}{(A \to B) \vee (B \to A)} \qquad \dfrac{}{(A \to B) \vee (B \to A)}}{(A \to B) \vee (B \to A)} \text{ contr}$$

Linearity in GLHN

$$\dfrac{\dfrac{\overline{A \to B}}{(A \to B) \vee (B \to A)}\ \vee\text{-i} \qquad \dfrac{}{(A \to B) \vee (B \to A)}}{(A \to B) \vee (B \to A)}\ \text{contr}$$

Linearity in GLHN

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B}}{A \to B}\ \to\text{-i}}{(A \to B) \vee (B \to A)}\ \vee\text{-i} \qquad \dfrac{}{(A \to B) \vee (B \to A)}}{(A \to B) \vee (B \to A)}\ \text{contr}$$

Linearity in GLHN

$$
\dfrac{
  \dfrac{
    \dfrac{\dfrac{}{B}}{A \to B}\;\to\text{-i}
  }{(A \to B) \vee (B \to A)}\;\vee\text{-i}
  \qquad
  \dfrac{
    \dfrac{}{B \to A}
  }{(A \to B) \vee (B \to A)}\;\vee\text{-i}
}{(A \to B) \vee (B \to A)}\;\text{contr}
$$

Linearity in GLHN

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B}}{A \to B}\ \to\text{-i}}{(A \to B) \vee (B \to A)}\ \vee\text{-i} \qquad \dfrac{\dfrac{\dfrac{}{A}}{B \to A}\ \to\text{-i}}{(A \to B) \vee (B \to A)}\ \vee\text{-i}}{(A \to B) \vee (B \to A)}\ \text{contr}$$

Linearity in GLHN

$$\dfrac{\dfrac{\dfrac{\dfrac{A}{B}\ \mathrm{com}^{x}_{A,B}}{A \to B}\ \to\text{-i}}{(A \to B) \vee (B \to A)}\ \vee\text{-i} \qquad \dfrac{\dfrac{\dfrac{B}{A}\ \mathrm{com}^{\bar{x}}_{B,A}}{B \to A}\ \to\text{-i}}{(A \to B) \vee (B \to A)}\ \vee\text{-i}}{(A \to B) \vee (B \to A)}\ \mathrm{contr}$$

Linearity in GLHN

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]}{B}\;\text{com}^{x}_{A,B}}{A \to B}\;\to\text{-i}}{(A \to B) \vee (B \to A)}\;\vee\text{-i} \qquad \dfrac{\dfrac{\dfrac{[B]}{A}\;\text{com}^{\bar{x}}_{B,A}}{B \to A}\;\to\text{-i}}{(A \to B) \vee (B \to A)}\;\vee\text{-i}}{(A \to B) \vee (B \to A)}\;\text{contr}$$

Simulation of ND rules

For every ND deduction rule there is a corresponding hyper rule operating on GLHNs.

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Example for binary rule:

- start with two GLHNs with disjoint sets of labels
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Easy and natural (in the ND sense) inductive construction of GLHNs is possible (first item on wishlist).

PROPERTIES

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GLHN deductions are closed under applying hyper rules.

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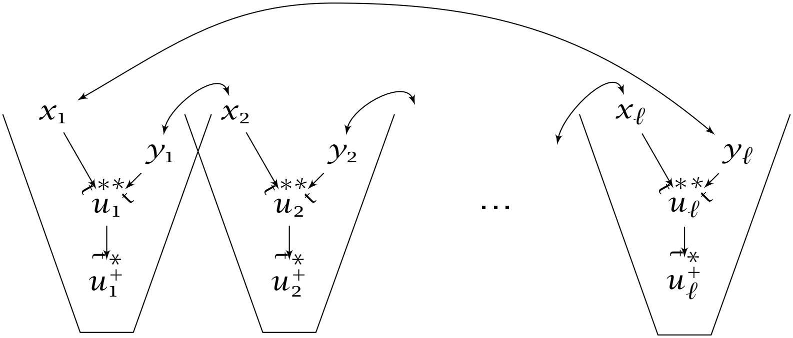
Corollary

GLHN sound and complete for GL

Core lemma

Chain lemma – in a GLHD the following figure cannot appear.



Normalisation

Conversions

- proof follows Troelstra/Schwichtenberg proof
- detour conversions, simplification conversion and permutation conversions as there, with cases for cut and split added
- branches and tracks
- double induction on cut-rank and ordinal sum of critical label sequences

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Theorem

Contraction, communication and splitting permutation conversions convert GLHN deductions into GLHN deductions.

RESULTS

Theorem (Normalisation)

GLHN admits (weak) normalisation. That is, there is a way to move all elimination rules above introduction rules by applying the above conversions.

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Theorem (Sub-formula property)

Let R be a normal hyper natural deduction with derived hypersequent $\mathcal{H}$. Then each formula in R is a subformula of a formula in $\mathcal{H}$.

Discussion of the system

- GLHN technical complex apparatus
- meta-theorems are complex
- deriving theorems, i.e., proving in GLHN is simple
- GLHN allows for normalisation via conversions with hope for computational interpretation
- (not the last word)

Returning to our wishlist

(semi) local

- ✔ construction of deductions:
  apply ND inspired rules to extend a HND deductions

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normalisation

- ✔ procedural normalisation via conversion steps

Further steps

- Extend GLHN to first order
- Reconsidering BCF system in the light of our procedural definition
- Develop term systems ("parallel λ") and establish Curry-Howard correspondences
- Investigate confluence of normalisation
- Connections to process algebra or other systems
- Extension to other hyper sequent systems

FURTHER STEPS

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- Develop term systems (“parallel λ”) and establish Curry-Howard correspondences
- Investigate confluence of normalisation
- Connections to process algebra or other systems
- Extension to other hyper sequent systems

Thanks for your attention!