Tense Operators on De Morgan posets

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Outline

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Introduction - tense operators on De Morgan posets

For Boolean algebras, the so-called tense operators were already introduced by Burgess. Tense operators express the quantifiers “it is always going to be the case that” and “it has always been the case that” and hence enable us to express the dimension of time in the logic. In this lecture we introduce tense operators on De Morgan posets.

A crucial problem concerning tense operators is their representation. Having a Boolean algebra with tense operators, it is well known that there exists a time frame such that each of these operators can be obtained by their construction for two-element Boolean algebra $\{0, 1\}$. We solved this problem with I. Chajda for tense operators on De Morgan posets.

A partial operator version of our results is valid as well (ISMVL 2016).

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Tense operators on $[0,1]$

Tense operators were used to express the dimension of time in logics.

- Let T be a time scale,
- then elements $f(t)$ from $[0,1]^T$ correspond to the evaluation of the validity of the formula f in time.

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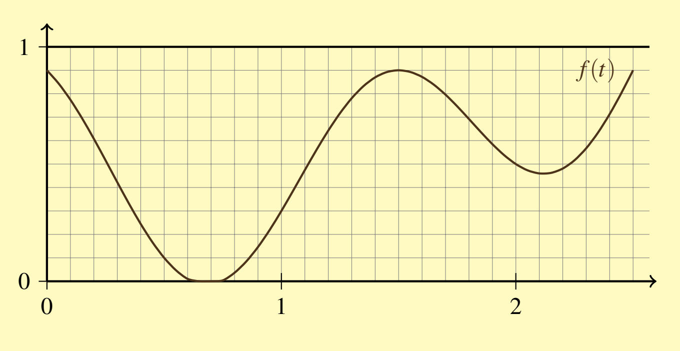

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On the time scale T we will introduce a relation $R \subseteq T^2$.

- $x\,R\,y$ means that the moment x is before the moment y.

Moreover, we introduce operators G a H on $[0,1]^T$ as follows:

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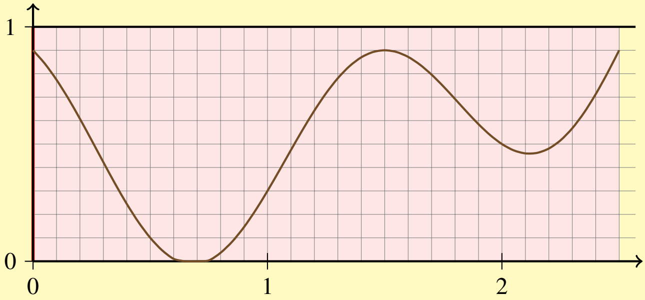

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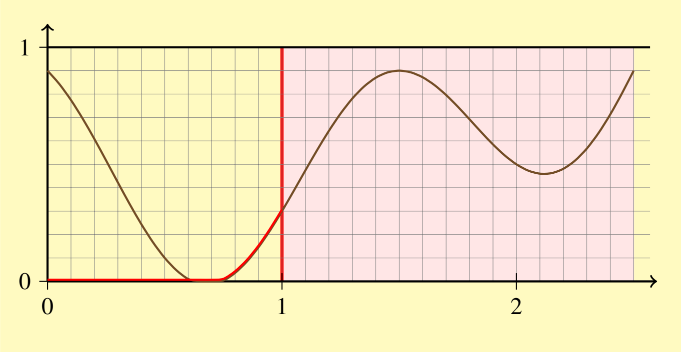

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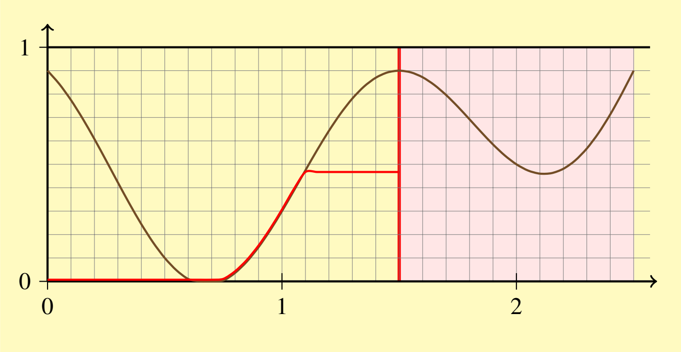

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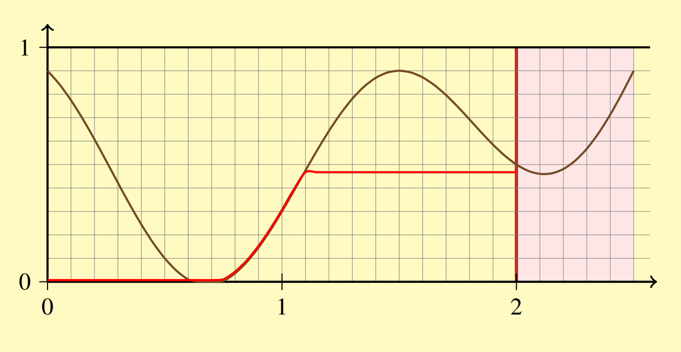

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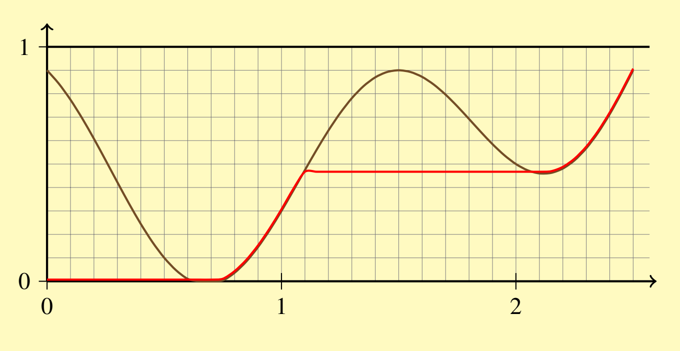

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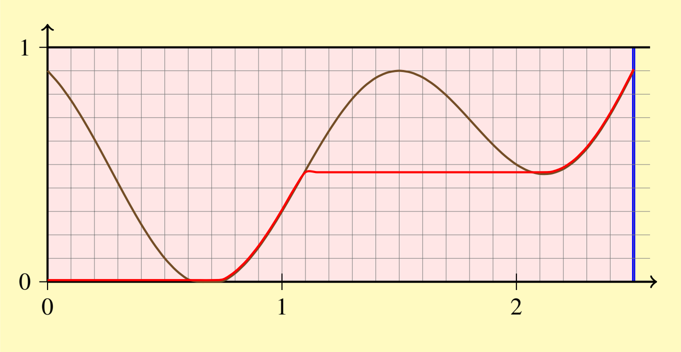

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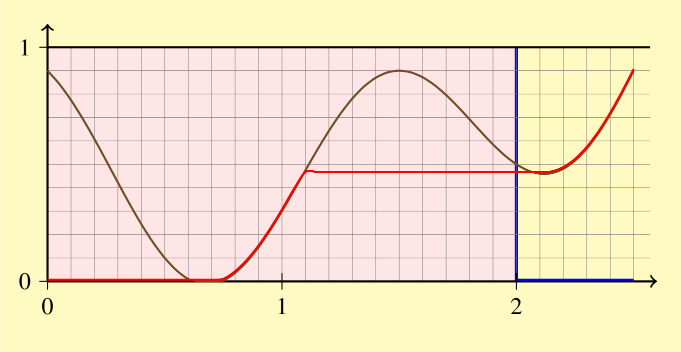

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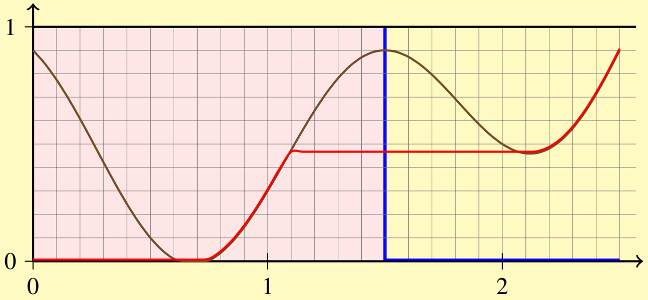

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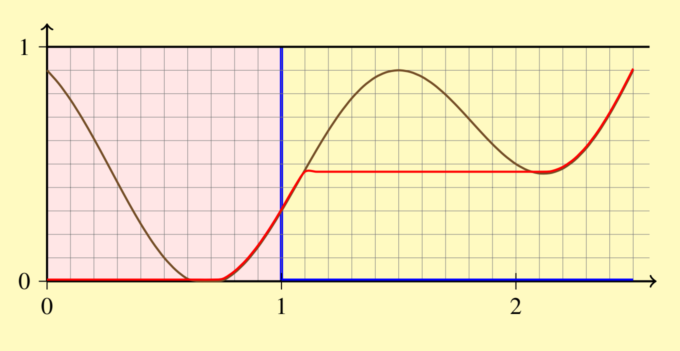

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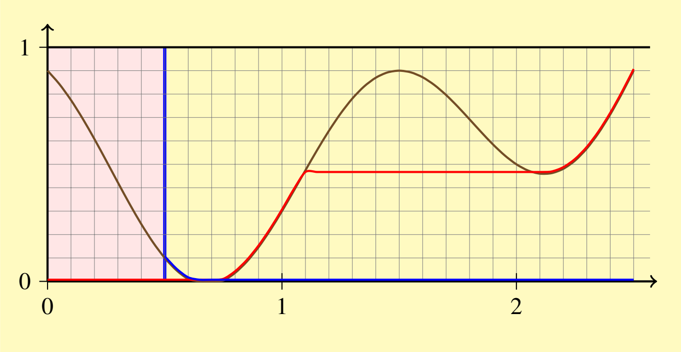

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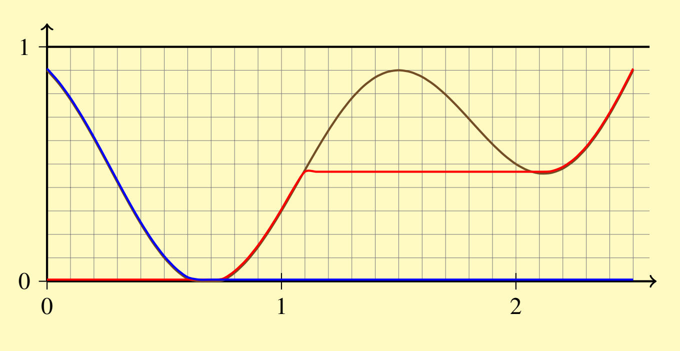

Outline

Let $\mathbf{A} = (A; \leq, ', 0, 1)$ be a bounded ordered set with an antitone involution (i.e., 0 is the least and 1 the greatest element). Then $\mathbf{A}$ is called a *De Morgan poset*. We notice that $0' = 1$, $1' = 0$ and $\mathbf{A}$ satisfies the so-called *De Morgan laws*:

$$(a \vee b)' = a' \wedge b' \quad \text{if } a \vee b \text{ exists} \quad \text{and}$$
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Basic definitions - tense operators on De Morgan posets

Definition

By the *tense De Morgan poset* is meant an algebra $(A; \leq, ', 0, 1, G, P, H, F)$ such that $\mathbf{A} = (A; \leq, ', 0, 1)$ is a **De Morgan poset**, (P, G) and (F, H) are **Galois connections** on A such that $G \leq F$, $H \leq P$, and for all $p, q \in A$,

$$F(p) = G(p')' \quad \text{and} \quad P(q) = H(q')'.$$

G, P, H and F are called *tense operators* on the tense De Morgan poset.

Note that the following holds:

$$x \leq GP(x) \text{ and } x \leq HF(x).$$

Let $(\mathbf{A}_1; G_1, P_1, H_1, F_1)$ and $(\mathbf{A}_2; G_2, P_2, H_2, F_2)$ be tense De Morgan posets. A *morphism of tense De Morgan posets* is a morphism of De Morgan posets $f : A_1 \to A_2$ which simultaneously commutes with the respective tense operators.

Basic definitions - tense operators on De Morgan posets

A *frame* is a triple (S, T, R) where S, T are non-empty sets, $R \subseteq S \times T$ such that

- for all $s \in S$ there is $t \in T$ such that $s\,R\,t$,
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If $S = T$ we say that the pair (T, R) is a *time frame.*

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Basic results – frames on complete De Morgan lattices

Theorem

Let $\mathbf{M}$ *be a complete De Morgan lattice,* (S,T,R) *be a frame,* $\widehat{G},\widehat{F}$ *be mappings from* M^T *into* M^S*, and* $\widehat{H},\widehat{P}$ *be mappings from* M^S *into* M^T *defined by*

$$\begin{aligned}\widehat{G}(p)(s) &= \bigwedge\{p(t) \mid t\in T, s\,R\,t\},\\ \widehat{F}(p)(s) &= \bigvee\{p(t) \mid t\in T, s\,R\,t\},\\ \widehat{H}(q)(t) &= \bigwedge\{q(s) \mid t\in T, s\,R\,t\},\\ \widehat{P}(q)(t) &= \bigvee\{q(s) \mid t\in T, s\,R\,t\}\end{aligned}$$

for all $p\in M^T$, $q\in M^S$*, and* $s\in S$, $t\in T$*. Then* $(\widehat{P},\widehat{G})$ *and* $(\widehat{F},\widehat{H})$ *are Galois connections such that* $\widehat{G}\le\widehat{F}$, $\widehat{H}\le\widehat{P}$*, and for all* $p\in M^T, q\in M^S$*,*

$$\widehat{F}(p)=\widehat{G}(p')' \quad \text{and} \quad \widehat{P}(q)=\widehat{H}(q')'.$$

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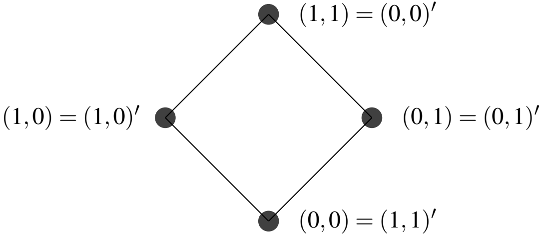
A canonical example of a De Morgan poset is the four-element De Morgan poset $\mathbf{M}_2$ depicted below.



Recall that the four-element De Morgan poset $\mathbf{M}_2$, considered as a distributive De Morgan lattice, generates the variety of all distributive De Morgan lattices. This result was the motivation for our study of the representation theorem of De Morgan posets.

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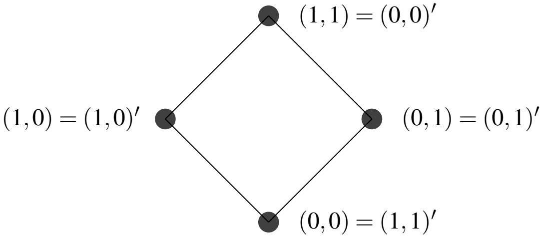

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Definition

Let $\mathbf{A}$ be a De Morgan poset and $\mathbf{M}$ be a complete De Morgan lattice. Let S be a set of morphisms of De Morgan posets from $\mathbf{A}$ to $\mathbf{M}$.

(a) S is called *order reflecting* or a *full set* if

$$((\forall s \in S)\, s(a) \leq s(b)) \implies a \leq b$$

for any elements $a, b \in A$.

(b) If S is an order-reflecting set then $\mathbf{A}$ is said to be *representable in* $\mathbf{M}$.

For any De Morgan poset $\mathbf{A} = (A; \leq, ', 0, 1)$, we let $T_{\mathbf{A}}^{\text{DMP}}$ denote a set of morphisms of De Morgan posets into the four-element De Morgan poset $\mathbf{M}_2$.

Proposition

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Let $\mathbf{A} = (A; \leq, ', 0, 1)$ *be a De Morgan poset. Then the map* $i_{\mathbf{A}} : A \to M_2^{T_{\mathbf{A}}^{DMP}}$ *given by* $i_{\mathbf{A}}(a)(s) = s(a)$ *for all* $a \in A$ *and all* $s \in T_{\mathbf{A}}^{DMP}$ *is an order-reflecting morphism of De Morgan posets such that* $i_{\mathbf{A}}(A)$ *is a De Morgan subposet of* $\mathbf{M}_2^{T_{\mathbf{A}}^{DMP}}$*.*

The construction of time frames and induced relations

We start with the construction of the relations R_G and R^P.

Let $\mathbf{A} = (A; \leq, 0, 1)$, $\mathbf{B} = (B; \leq, 0, 1)$, and $\mathbf{C} = (C; \leq, 0, 1)$ be bounded posets such that $\mathbf{C}$ is non-trivial, S a set of bounded poset morphisms from $\mathbf{A}$ to $\mathbf{C}$, and T a set of bounded poset morphisms from $\mathbf{B}$ to $\mathbf{C}$. Let $P : A \to B$ and $G : B \to A$ be morphisms of posets. Let us define the relations

$$R_G = \{(s,t) \in S \times T \mid (\forall b \in B)(s(G(b)) \leq t(b))\} \qquad (\dagger)$$

and

$$R^P = \{(s,t) \in S \times T \mid (\forall a \in A)(s(a) \leq t(P(a)))\}. \qquad (\dagger\dagger)$$

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The construction of time frames and induced relations

Motivated by the construction of tense De Morgan posets assume, moreover, that we have morphisms of posets $F : B \to A$ and $H : A \to B$. Then we obtain from (†) and (††) relations $R_H = \{(t,s) \in T \times S \mid (\forall a \in A)(t(H(a)) \leq s(a))\}$ and $R^F = \{(t,s) \in T \times S \mid (\forall b \in B)(t(b) \leq s(F(b)))\}$.

Combining R_G with R_H and R^F we obtain the following relations

$$R_{G,H} = \{(s,t) \in S \times T \mid \quad (\forall b \in B)(s(G(b)) \leq t(b)) \text{ and } (\forall a \in A)(t(H(a)) \leq s(a))\} \qquad (\ddagger)$$

and

$$R^F_G = \{(s,t) \in S \times T \mid (\forall b \in B)(s(G(b)) \leq t(b) \leq s(F(b)))\}. \qquad (\ddagger\ddagger)$$

The relations R_G, R^P, $R_{G,H}$, and R^F_G on $S \times T$ will be called the *G-induced relation by* $\mathbf{C}$, *P-induced relation by* $\mathbf{C}$, *(G,H)-induced relation by* $\mathbf{C}$, and *(G,F)-induced relation by* $\mathbf{C}$, respectively.

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$$R_{G,H} = \{(s,t) \in S \times T \mid \quad \begin{array}{l}(\forall b \in B)(s(G(b)) \leq t(b)) \text{ and} \\ (\forall a \in A)(t(H(a)) \leq s(a))\}\end{array} \tag{‡}$$

and

$$R^F_G = \{(s,t) \in S \times T \mid (\forall b \in B)(s(G(b)) \leq t(b) \leq s(F(b)))\}. \tag{‡‡}$$

The relations R_G, R^P, $R_{G,H}$, and R^F_G on $S \times T$ will be called the *G-induced relation by* $\mathbf{C}$, *P-induced relation by* $\mathbf{C}$, *(G,H)-induced relation by* $\mathbf{C}$, and *(G,F)-induced relation by* $\mathbf{C}$, respectively.

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and

$$R_G^F = \{(s,t) \in S \times T \mid (\forall b \in B)(s(G(b)) \leq t(b) \leq s(F(b)))\}. \tag{‡‡}$$

The relations R_G, R^P, $R_{G,H}$, and R_G^F on $S \times T$ will be called the *G-induced relation by* $\mathbf{C}$, *P-induced relation by* $\mathbf{C}$, *(G,H)-induced relation by* $\mathbf{C}$, and *(G,F)-induced relation by* $\mathbf{C}$, respectively.

Theorem

Let $\mathbf{A} = (A; \leq, ', 0, 1)$ *and* $\mathbf{B} = (B; \leq, ', 0, 1)$ *be De Morgan posets,* $F, G \colon B \to A$ *be order-preserving mappings such that* $G(1) = F(1) = 1, G(0) = F(0) = 0$, $G \leq F$ *and* $F = {}' \circ G \circ {}'$. *Let* R_G *and* R_G^F *be the* G*-induced and* (G, F)*-induced relations on* $T_{\mathbf{A}}^{DMP} \times T_{\mathbf{B}}^{DMP}$ *by* $\mathbf{M}_2$*, respectively, and let* $\widehat{G}$ *and* $\widehat{F}$ *be constructed by means of the frame* $(T_{\mathbf{A}}^{DMP}, T_{\mathbf{B}}^{DMP}, R_G^F)$*.*

Then $R_G = R_G^F = (R^F)^{-1}$ *and the mappings* $i_{\mathbf{A}}$*,* $i_{\mathbf{B}}$ *are order-reflecting morphisms of De Morgan posets into the complete De Morgan lattices* $\mathbf{M}_2^{T_{\mathbf{A}}^{DMP}}$ *and* $\mathbf{M}_2^{T_{\mathbf{B}}^{DMP}}$ *such that the following diagrams commute:*

$$\begin{array}{ccc} B & \xrightarrow{\quad F \quad} & A \\ \downarrow{\scriptstyle i_{\mathbf{B}}} & & \downarrow{\scriptstyle i_{\mathbf{A}}} \\ \mathbf{M}_2^{T_{\mathbf{B}}^{DMP}} & \xrightarrow[\quad \widehat{F} \quad]{} & \mathbf{M}_2^{T_{\mathbf{A}}^{DMP}} \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\quad G \quad} & A \\ \downarrow{\scriptstyle i_{\mathbf{B}}} & & \downarrow{\scriptstyle i_{\mathbf{A}}} \\ \mathbf{M}_2^{T_{\mathbf{B}}^{DMP}} & \xrightarrow[\quad \widehat{G} \quad]{} & \mathbf{M}_2^{T_{\mathbf{A}}^{DMP}} \end{array}$$

Set representation theorem for tense De Morgan posets

Theorem

Let $(\mathbf{A};G,P,H,F)$ *be a tense De Morgan poset and* R_G *the* G*-induced relation on* $T_{\mathbf{A}}^{DMP}$ *by* $\mathbf{M}_2$*.*

Then the map $i_{\mathbf{A}}^{T_{\mathbf{A}}^{DMP}}$ *is an order-reflecting morphism of tense De Morgan posets into the complete tense De Morgan poset* $(\mathbf{M}_2^{T_{\mathbf{A}}^{DMP}};\widehat{G},\widehat{P},\widehat{H},\widehat{F})$ *constructed by the time frame* $(T_{\mathbf{A}}^{DMP},R_G)$*.*

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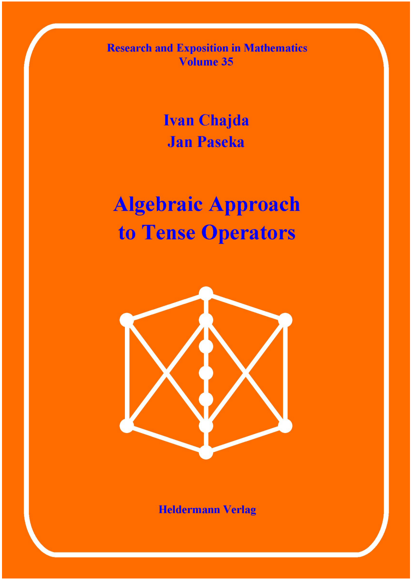

Thank you for your attention.