Completeness properties in protoalgebraic logics

Petr Cintula[1] Carles Noguera[2]

[1]Institute of Computer Science
Czech Academy of Sciences

[2]Institute of Information Theory and Automation
Czech Academy of Sciences

Aim of the talk

Logic: Syntax ⇔ Semantics

Completeness theorems

- Study them from the point of view of abstract algebraic logic
- Consider different forms of completeness for arbitrary classes of (reduced) matrices
- Give useful characterizations
- Understand the role of connectives (implication and disjunction) in completeness theorems

Level of generality: protoalgebraic logics

The precedents

1. Janusz Czelakowski. *Protoalgebraic Logics*. Trends in Logic, vol 10, Dordrecht, Kluwer, 2001.
2. Petr Cintula, Francesc Esteva, Joan Gispert, Lluís Godo, Franco Montagna, Carles Noguera. Distinguished algebraic semantics for t-norm based fuzzy logics: Methods and algebraic equivalencies, *Annals of Pure and Applied Logic* 160, 53–81, 2009.
3. Petr Cintula and Carles Noguera. A General Framework for Mathematical Fuzzy Logic. In *Handbook of Mathematical Fuzzy Logic – Volume 1*. Studies in Logic, Mathematical Logic and Foundations, vol. 37, London, College Publications, pp. 103–207, 2011.

What is a logic (in AAL)

Var: an infinite set of propositional variables

$\mathcal{L}$: an arbitrary type

$\boldsymbol{Fm}_{\mathcal{L}}$: the absolutely free $\mathcal{L}$-algebra with generators Var

elements of $Fm_{\mathcal{L}}$ are called *$\mathcal{L}$-formulae*

A logic L is a relation between sets of $\mathcal{L}$-formulae and $\mathcal{L}$-formulae s.t.:

we write '$\Gamma \vdash_{\mathrm{L}} \varphi$' instead of '$\langle \Gamma, \varphi \rangle \in \mathrm{L}$'

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Monotonicity)
- If $\Delta \vdash_{\mathrm{L}} \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Cut)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each substitution σ. (Structurality)

A logic L is finitary if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ s.t. $\Gamma' \vdash_{\mathrm{L}} \varphi$

What is a completeness theorem? – 1

Matrix model: $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and $F \subseteq A$, such that

if $\Gamma \vdash_{\mathrm{L}} \varphi$, then for each $\boldsymbol{A}$-evaluation e such that $e[\Gamma] \subseteq F$, we have $e(\varphi) \in F$

Class of all models: $\mathbf{MOD}(\mathrm{L})$

1st completeness theorem: for each Γ and φ, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbf{MOD}(\mathrm{L})} \varphi$

Reduced model: $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ such that $\Omega_{\boldsymbol{A}}(F) = Id_{\boldsymbol{A}}$

Class of all reduced models: $\mathbf{MOD}^*(\mathrm{L})$

2nd completeness theorem: for each Γ and φ, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi$

What is a completeness theorem? – 2

3rd completeness theorem: If L is finitary, then for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathrm{L}} \varphi \text{ iff } \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}} \varphi$$

Definition

Let L be a logic and $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. We say that L has the property of:

- Strong $\mathbb{K}$-completeness, $\mathrm{S}\mathbb{K}\mathrm{C}$ for short, when for every set of formulae $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$
- Finite strong $\mathbb{K}$-completeness, $\mathrm{FS}\mathbb{K}\mathrm{C}$ for short, when for every finite set of formulae $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$
- $\mathbb{K}$-completeness, $\mathbb{K}\mathrm{C}$ for short, when for every formula φ: $\vdash_{\mathrm{L}} \varphi$ iff $\models_{\mathbb{K}} \varphi$

What is a protoalgebraic logic?

Let $\overrightarrow{r}$ be a sequence of atoms and $\Rightarrow(p, q, \overrightarrow{r}) \subseteq Fm_{\mathcal{L}}$

Convention: given formulae φ and ψ, we set

$$\varphi \Rightarrow \psi \quad = \quad \bigcup \{\Rightarrow(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{<\omega}\}$$

A logic is protoalgebraic if it has a weak p-implication, i.e., a set $\Rightarrow$ s.t.:

(R)	$\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$
(MP)	$\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$
(T)	$\varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi$
(sCng)	$\varphi \Rightarrow \psi, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots, \varphi, \dots, \chi_n) \Rightarrow c(\chi_1, \dots, \psi, \dots, \chi_n)$ for each $\langle c, n \rangle \in \mathcal{L}$ and $i \leq n$

Pre(ordered) matrices

Consider $\mathbf{A} = \langle A, F\rangle \in \mathbf{MOD}(\mathrm{L})$; then the relation $a \leq^{\Rightarrow}_{\mathbf{A}} b$ is a preorder:

$$a \leq^{\Rightarrow}_{\mathbf{A}} b \qquad \text{iff} \qquad a \Rightarrow^{\boldsymbol{A}} b \subseteq F$$

$\leq^{\Rightarrow}_{\mathbf{A}}$ is an order iff $\mathbf{A} \in \mathbf{MOD}^{*}(\mathrm{L})$

Disjunction

A connective $\vee$ is a lattice-disjunction in L if for each $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}}$:

$$\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi \vee \psi \qquad \vdash_{\mathrm{L}} \psi \Rightarrow \varphi \vee \psi \qquad \varphi \Rightarrow \chi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \vee \psi \Rightarrow \chi$$

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi}$$

Completeness w.r.t. linearly ordered matrices

A matrix $\mathbf{A}$ is linear, $\mathbf{A} \in \mathbf{MOD}^{\ell}(\mathrm{L})$, if $\leq^{\Rightarrow}_{\mathbf{A}}$ is a linear order.

Theories are deductively closed sets of L

A theory T is linear if for any pair φ, ψ we have: $\varphi \Rightarrow \psi \subseteq T$ or $\psi \Rightarrow \varphi \subseteq T$

Theorem

Let L be a protoalgebraic logic. TFAE:

1. L *is semilinear, i.e.,* $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\ell}(\mathrm{L})}$
2. *Any theory* $T \nvdash_{\mathrm{L}} \varphi$ *can be extended into a linear theory* $T' \nvdash_{\mathrm{L}} \varphi$
3. L *has the* IPEP *and the semilinearity property, i.e., for each* $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}}$:

$$\frac{\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}$$

Completeness w.r.t. densely ordered matrices

A matrix $\mathbf{A}$ is densely linear, $\mathbf{A} \in \mathbf{MOD}^{\delta}(\mathrm{L})$, if $\leq_{\mathbf{A}}^{\Rightarrow}$ is a dense linear order.

A linear theory T is dense if for any pair φ, ψ if $\psi \Rightarrow \varphi \notin T$ there is χ s.t. $\chi \Rightarrow \varphi \notin T$ and $\psi \Rightarrow \chi \notin T$

Theorem

Let L *be a protoalgebraic logic. TFAE:*

1. L *is dense complete, i.e.,* $\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^{\delta}(\mathrm{L})}$
2. *Any set of formulae* $\Gamma \nvdash_{\mathrm{L}} \varphi$ *with infinitely many unused variables can be extended into a dense theory* $T' \nvdash_{\mathrm{L}} \varphi$

If L *is finitary,* $\Rightarrow$ *finite and parameter-free,* $\vee$ *a lattice-disjunction, we can add*

3. *Countable chains in* $\mathbf{MOD}^{\ell}(\mathrm{L})$ *can be embedded into dense ones*
4. L *has the density property, i.e., for each* $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}}$ *and any variable* p *not occurring in* $\Gamma \cup \{\varphi, \psi, \chi\}$*:*

$$\frac{\Gamma \vdash_{\mathrm{L}} (\varphi \Rightarrow p) \vee (p \Rightarrow \psi) \vee \chi}{\Gamma \vdash_{\mathrm{L}} (\varphi \Rightarrow \psi) \vee \chi}$$

Completeness w.r.t. arbitrary matrices – 1

General assumptions:

- L is a finitary protoalgebraic logic
- $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$

Theorem (Czelakowski, Th. 1.3.10)

The following are equivalent:

1. L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$
2. *Each* RSI *Lindenbaum–Tarski matrix is in* $\mathbf{H}_{\mathrm{S}}\mathbf{S}(\mathbb{K}^+)$

Many particular completeness theorems are proved by giving an embedding into the class $\mathbb{K}$.

Can we have a characterization in terms of embeddings?

Completeness w.r.t. arbitrary matrices – 2

A set of formulae Ψ is directed if for each $\varphi, \psi \in \Psi$ there is $\chi \in \Psi$ such that $\vdash_{\mathrm{L}} \varphi \Rightarrow \chi$ and $\vdash_{\mathrm{L}} \psi \Rightarrow \chi$ (we call χ an upper bound of φ and ψ).

Lemma

If L has the $\mathrm{S}\mathbb{K}\mathrm{C}$, then for every set of formulae Γ and every directed set of formulae Ψ the following are equivalent:

- $\Gamma \nvdash_{\mathrm{L}} \psi$ *for each* $\psi \in \Psi$.
- *There is a matrix* $\langle \boldsymbol{A}, F \rangle \in \mathbb{K}$ *and an* $\boldsymbol{A}$*-evaluation* e *such that* $e[\Gamma] \subseteq F$ *and* $e[\Psi] \cap F = \emptyset$.

Idea of the proof

1. Assume v does not appear in $\Gamma \cup \Psi$.
2. $\Gamma' = \Gamma \cup \{\psi \Rightarrow v \mid \psi \in \Psi\}$.
3. Assume that $\Gamma' \vdash_{\mathrm{L}} v$. There are finite $\hat{\Gamma} \subseteq \Gamma$ and $\hat{\Psi} \subseteq \Psi$ s.t. $\hat{\Gamma} \cup \{\psi \Rightarrow v \mid \psi \in \hat{\Psi}\} \vdash_{\mathrm{L}} v$. Let $\delta \in \Psi$ be an upper bound of $\hat{\Psi}$.
4. Take the substitution σ such that $\sigma(v) = \delta$ and $\sigma(p) = p$ for each $p \neq v$.
5. $\hat{\Gamma} \cup \{\sigma(\psi \Rightarrow v) \mid \psi \in \hat{\Psi}\} \vdash_{\mathrm{L}} \delta$.
6. Since $\sigma(\psi \Rightarrow v) \subseteq \sigma(\psi) \Rightarrow \sigma(v) = \psi \Rightarrow \delta$, we have $\hat{\Gamma} \cup \{\psi \Rightarrow \delta \mid \psi \in \hat{\Psi}\} \vdash_{\mathrm{L}} \delta$ and, hence, $\hat{\Gamma} \vdash_{\mathrm{L}} \delta$ –a contradiction!
7. Thus, by the $\mathrm{S}\mathbb{K}\mathrm{C}$, there are $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and e s.t. $e[\Gamma'] \subseteq F$ and $e(v) \notin F$. Thus $e[\Psi] \cap F = \emptyset$ (if $e(\psi) \in F$ for some $\psi \in \Psi$ then, since $e[\Gamma'] \subseteq F$, we would obtain $e(v) \in F$ –a contradiction!).
8. If $\Gamma \cup \Psi$ uses all variables, we use a Hilbert hotel argument first.

Completeness w.r.t. arbitrary matrices – 2

Theorem

If L *has a lattice disjunction* $\vee$ *and* $\mathbf{S}(\mathbb{K}) \subseteq \mathbf{MOD}^*(\mathrm{L})$*, the following are equivalent:*

(i) L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$.

(ii) *For each prime theory* T*, there is a strict homomorphism from* $\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle$ *into some member of* $\mathbb{K}$.

(iii) *Every non-trivial countable member of* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ *is embeddable by a strict homomorphism into some member of* $\mathbb{K}$.

(iv) *Every countable member of* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ *is embeddable by a strict homomorphism into some member of* $\mathbb{K}$.

Idea of the proof

1. T be a prime theory. $\Psi = Fm_{\mathcal{L}} \setminus T$ is directed: if $\varphi, \psi \in \Psi$, then $\varphi \vee \psi \in \Psi$ and, since $\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi \vee \psi$ and $\vdash_{\mathrm{L}} \psi \Rightarrow \varphi \vee \psi$, it is an upper bound.
2. By the lemma, there is $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and e such that $e[T] \subseteq F$ and $e[\Psi] \cap F = \emptyset$. Thus we have a strict homomorphism $e \colon \langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle \to \langle \boldsymbol{A}, F\rangle$.
3. Each non-trivial reduced countable RFSI model is isomorphic to $\langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T)\rangle$ for a prime theory T. Thus, there is $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and a strict homomorphism $e \colon \langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle \to \langle \boldsymbol{A}, F\rangle$.
4. Let $\langle \boldsymbol{B}, G\rangle$ be the submatrix of $\langle \boldsymbol{A}, F\rangle$ given by the image of e.
5. By assumption, $\langle \boldsymbol{B}, G\rangle \in \mathbf{MOD}^*(\mathrm{L})$.
6. $f \colon \langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T)\rangle \to \langle \boldsymbol{B}, G\rangle$ defined as $f(\varphi/\Omega(T)) = e(\varphi)$ is the desired embedding.

Characterization of the finite completeness

$\langle \boldsymbol{A}, F\rangle$ is partially embeddable into $\langle \boldsymbol{B}, G\rangle$ if for each finite $X \subseteq A$ there is a one-to-one mapping $f\colon X \to B$ such that:

- it sends elements of F to G
- for each $c \in \mathcal{L}$ and elements $a_1, \dots, a_n \in X$:

  if $c^{\boldsymbol{A}}(a_1, \dots, a_n) \in X$, then $f(c^{\boldsymbol{A}}(a_1, \dots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \dots, f(a_n))$.

Theorem

If L has a finite language and a lattice disjunction $\vee$, and it is finitely equivalential, the following are equivalent:

(i) L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$.

(ii) *Every trivial member of* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ *is partially embeddable into* $\mathbb{K}$.

(iii) *Every non-trivial countable member of* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ *is part. embeddable into* $\mathbb{K}$.

Characterization of the last completeness property

Theorem

Let L *be a* *protoalgebraic* *logic and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. *Then:*

(i) L *has the* $\mathbb{K}\mathrm{C}$

(ii) $\mathbf{MOD}^*(\mathrm{L}) \subseteq \mathbf{HSP}(\mathbb{K})$

Thank you for your attention!