Filtrations for many-valued modal logics.

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Filtrations

Recall that Σ is called *subformula-closed* if, whenever $\varphi \in \Sigma$ and ψ is a subformula of φ, then $\psi \in \Sigma$.

Let $\mathfrak{M} = (W, \mathrm{R}, V)$ be a model for the basic modal language and Σ a subformula closed set of formulas. Let $\leftrightsquigarrow_\Sigma \subseteq W \times W$ be defined by

$w \leftrightsquigarrow_\Sigma v \iff$ for all $\varphi \in \Sigma$ we have that $\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{M}, v \Vdash \varphi$.

Then $\leftrightsquigarrow_\Sigma$ is an equivalence relation with equivalences classes denoted by $[w]_\Sigma$. Let $W_\Sigma = \{[w]_\Sigma \mid w \in W\}$.

Suppose $\mathfrak{M}^f_\Sigma$ is any model (W^f, R^f, V^f) such that:

(i) $W^f = W_\Sigma$.

(ii) If $\mathrm{R}\, wv$, then $R^f[w][v]$.

(iii) If $\mathrm{R}^f[w][v]$, then for all $\Diamond\varphi \in \Sigma$, if $\mathfrak{M}, v \Vdash \varphi$, then $\mathfrak{M}, w \Vdash \Diamond\varphi$.

(iv) $V^f(p) = \{[w] \mid \mathfrak{M}, w \Vdash p\}$, for all proposition letters p in Σ.

Then $\mathfrak{M}^f_\Sigma$ is called a *filtration of $\mathfrak{M}$ through Σ.*

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Then $\mathfrak{M}^f_\Sigma$ is called a *filtration of* $\mathfrak{M}$ *through* Σ.

Filtration theorem

Theorem

Let $\mathfrak{M}^f$ be a filtration of $\mathfrak{M}$ through a subformula closed set Σ. Then, for all formulas $\varphi \in \Sigma$, and all nodes w in $\mathfrak{M}$, we have that

$$\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{M}^f, [w] \Vdash \varphi.$$

Theorem

Let φ be a basic modal formula. If φ is satisfiable, then it is satisfiable on a finite model. Moreover, it is satisfiable on a finite model containing at most 2^m nodes, where m is the number of subformulas of φ.

The language

The formulas of the *basic many-valued modal language* over a denumerably infinite set of proposition letters Φ are given by the following recursive definition:

$$\varphi := \bot \mid p \mid p \vee q \mid p \wedge q \mid p \rightarrow q \mid \Diamond p \mid \Box p.$$

Kripke semantics

Let $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$ be a Heyting algebra.

Kripke semantics

Definition

A *many-valued (Kripke) frame* over $\mathbf{A}$ ($\mathbf{A}$-frame) for the basic many-valued modal language is a triple $\mathfrak{F} = (W, D, B)$ with a nonempty universe W and many-valued accessibility relations $D : W \times W \to \mathbf{A}$ and $B : W \times W \to \mathbf{A}$.

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The valuation can be extended to all formulas. In particular,

$$V(w, \Diamond\varphi) = \bigvee\{D(w, v) \wedge V(v, \varphi) \mid v \in W\} \text{ and}$$

$$V(w, \Box\varphi) = \bigwedge\{B(w, v) \to V(v, \varphi) \mid v \in W\}.$$

Degrees of truth

Let $a \in \mathbf{A}$, then a formula φ is said to be *a-true* in an **A**-model $\mathfrak{M}$ at $w \in W$, denoted by $\mathfrak{M}, w \Vdash_a \varphi$, if $V(w, \varphi) \geq a$.

Equivalence classes

The equivalence relation with respect to a subformula-closed set of formulas, Σ, on a many-valued model $\mathfrak{M}$ over $\mathbf{A}$:

$$\text{for } w, v \in W \text{ and } \varphi \in \Sigma,\ w \leftrightsquigarrow^{\mathbf{A}}_{\Sigma} v \text{ if, and only if,}$$
$$\mathfrak{M}, w \Vdash_a \varphi \iff \mathfrak{M}, v \Vdash_a \varphi \text{ for all } a \in \mathbf{A}.$$

Let $[w]^{\mathbf{A}}_{\Sigma}$ denote the equivalence class of $w \in W$. If Σ and $\mathbf{A}$ are clear from the context we just write $[w]$.
Let $W^{\mathbf{A}}_{\Sigma} = \{[w] \mid w \in W\}$.

Filtrations for many-valued Kripke models

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$(R1)$ Let $a \in \mathbf{A}$. If $\mathrm{D}\, wv \geq a$, then $\mathrm{D}_f^{(\Sigma,\mathbf{A})}[w][v] \geq a$.

$(R2)$ Let $a_1, a_2 \in \mathbf{A}$. If $\mathrm{D}_f^{(\Sigma,\mathbf{A})}[w][v] \geq a_1$, then for every $\Diamond\varphi \in \Sigma$, if $\mathfrak{M}, v \Vdash_{a_2} \varphi$, then $\mathfrak{M}, w \Vdash_{a_1 \wedge a_2} \Diamond\varphi$.

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($R3$) Let $a \in \mathbf{A}$. If $\mathsf{B}\, wv \geq a$, then $\mathsf{B}_f^{(\Sigma,\mathbf{A})}[w][v] \geq a$.

($R4$) Let $a_1, a_2 \in \mathbf{A}$. If $\mathsf{B}_f^{(\Sigma,\mathbf{A})}[w][v] \geq a_1$, then for every $\Box\varphi \in \Sigma$, if $\mathfrak{M}, w \Vdash_{a_2} \Box\varphi$, then $\mathfrak{M}, v \Vdash_{a_1 \wedge a_2} \varphi$.

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(V) $V_f^{\mathbf{A}}([w], p) = V(w, p)$ for all $p \in \Sigma$.

Then $\mathfrak{M}_f^{(\Sigma,\mathbf{A})}$ is called a *filtration of* $\mathfrak{M}$ *through* Σ *over* $\mathbf{A}$.

The Many-valued filtration theorem

Lemma

Let Σ be a subformula closed set of formulas and suppose $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, \mathrm{D}_f^{(\Sigma,\mathbf{A})}, \mathrm{B}_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$ is a filtration of $\mathfrak{M} = (W, D, B, V)$ through Σ over $\mathbf{A}$. For all $\varphi \in \Sigma$ and all nodes $w \in W$ we have $V(w, \varphi) = V_\Sigma^{\mathbf{A}}([w], \varphi)$

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Proof (Sketch).

Induction on formulas.

- If $\varphi := \Diamond\psi$, then $V(w, \Diamond\psi) \leq V_\Sigma^{\mathbf{A}}([w], \Diamond\psi)$ follows from $(R1)$ and $V(w, \Diamond\psi) \geq V_\Sigma^{\mathbf{A}}([w], \Diamond\psi)$ from $(R2)$.
- If $\varphi := \Box\psi$, then $V(w, \Box\psi) \geq V_\Sigma^{\mathbf{A}}([w], \Diamond\psi)$ follows from $(R3)$ and $V(w, \Box\psi) \geq V_\Sigma^{\mathbf{A}}([w], \Box\psi)$ from $(R4)$.

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- If $\varphi := \Box\psi$, then $V(w, \Box\psi) \geq V_\Sigma^{\mathbf{A}}([w], \Diamond\psi)$ follows from $(R3)$ and $V(w, \Box\psi) \geq V_\Sigma^{\mathbf{A}}([w], \Box\psi)$ from $(R4)$.

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Theorem

Let $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, \mathrm{D}_f^{(\Sigma,\mathbf{A})}, \mathrm{B}_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$ be a filtration of $\mathfrak{M}$ through a subformula closed set Σ over $\mathbf{A}$. Then, for all formulas $\varphi \in \Sigma$, all states w in $\mathfrak{M}$ and any truth value $a \neq 0$ in $\mathbf{A}$, we have that

$$\mathfrak{M}, w \Vdash_a \varphi \iff \mathfrak{M}_f^{(\Sigma,\mathbf{A})}, [w]_\Sigma^{\mathbf{A}} \Vdash_a \varphi.$$

Finite model property

Theorem

Let φ be a basic many-valued modal formula. If φ is satisfiable in an $\mathbf{A}$-model where $\mathbf{A}$ is a finite *Heyting algebra, then φ is satisfiable on a finite model containing at most 2^m states, where m is the product of $|\varphi|$ and $|\mathbf{A}|$.*

The smallest filtration

For the basic modal language the *smallest filtration* is obtained by using the following rule to describe the accessibility relation:

$$
\begin{aligned}
& R^s[w][v] \\
\Longleftrightarrow\ & \text{there exists } w' \in [w], \text{ there exists } v' \in [w] \\
& \text{such that } \mathrm{R}\, w' v'.
\end{aligned}
$$

The smallest filtration

$$D^{\mathbf{A}}_s[w][v] = \bigvee \{Dw'v' \mid w' \in [w], v' \in [v]\}$$

$$B^{\mathbf{A}}_s[w][v] = \bigvee \{Bw'v' \mid w' \in [w], v' \in [v]\}$$

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Proposition

Suppose Σ is subformula closed and let $\mathfrak{M}^{(\Sigma,\mathbf{A})}_s = (W^{\mathbf{A}}_\Sigma, \mathrm{D}^{(\Sigma,\mathbf{A})}_s, \mathrm{B}^{(\Sigma,\mathbf{A})}_s, V^{\mathbf{A}}_\Sigma)$, where $D^{\mathbf{A}}_s$ and $B^{\mathbf{A}}_s$ are obtained as described above. Then $\mathfrak{M}^{(\Sigma,\mathbf{A})}_s$ is a filtration of $\mathfrak{M}$ through Σ over $\mathbf{A}$.
Moreover, if $\mathfrak{M}^{(\Sigma,\mathbf{A})}_f = (W^{\mathbf{A}}_\Sigma, \mathrm{D}^{(\Sigma,\mathbf{A})}_f, \mathrm{B}^{(\Sigma,\mathbf{A})}_f, V^{\mathbf{A}}_\Sigma)$ is any filtration of $\mathfrak{M}$ through Σ over $\mathbf{A}$, then for all $[w], [v] \in W^{\mathbf{A}}_\Sigma$, we have that

$$D^{\mathbf{A}}_s[w][v] \leq D^{\mathbf{A}}_f[w][v] \quad \text{and} \quad B^{\mathbf{A}}_s[w][v] \leq B^{\mathbf{A}}_f[w][v]$$

The largest filtration

For the basic modal language the *largest filtration* is obtained by using the following rule to describe the accessibility relation:

$$
\begin{aligned}
& R^{\ell}[w][v] \\
\Longleftrightarrow\ & \text{for all formulas } \Diamond\varphi \in \Sigma : \\
& \mathfrak{M}, v \Vdash \varphi \text{ implies } \mathfrak{M}, w \Vdash \Diamond\varphi.
\end{aligned}
$$

The largest filtration

$$D_\ell^{\mathbf{A}}[w][v] = \bigwedge \{V(v,\varphi) \rightarrow V(w,\Diamond\varphi) \mid \Diamond\varphi \in \Sigma\}$$

$$B_\ell^{\mathbf{A}}[w][v] = \bigwedge \{V(w,\Box\varphi) \rightarrow V(v,\varphi) \mid \Box\varphi \in \Sigma\}$$

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Proposition

Suppose Σ is subformula closed and let $\mathfrak{M}_\ell^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, \mathrm{D}_\ell^{(\Sigma,\mathbf{A})}, \mathrm{B}_\ell^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$ where $D_\ell^{\mathbf{A}}$ and $B_\ell^{\mathbf{A}}$ are obtained as described above. Then $\mathfrak{M}_\ell^{(\Sigma,\mathbf{A})}$ is a filtration of $\mathfrak{M}$ through Σ over $\mathbf{A}$.
Moreover, if $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, \mathrm{D}_f^{(\Sigma,\mathbf{A})}, \mathrm{B}_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$ is any filtration of $\mathfrak{M}$ through Σ over $\mathbf{A}$, then for all $[w],[v] \in W_\Sigma^{\mathbf{A}}$, we have that

$$D_\ell^{\mathbf{A}}[w][v] \geq D_f^{\mathbf{A}}[w][v] \quad \text{and} \quad B_\ell^{\mathbf{A}}[w][v] \geq B_f^{\mathbf{A}}[w][v]$$

Many-valued transitivity

Fitting:[1]

Definition

Let $\mathfrak{F} = (W, D, B)$ be an **A**-frame. Then D and B are said to be *transitive* if for all $u, v, w \in W$ we have that

$$\mathrm{D}\,wv \wedge \mathrm{D}\,vu \leq \mathrm{D}\,wu \quad \text{and} \quad \mathrm{B}\,wv \wedge \mathrm{B}\,vu \leq \mathrm{B}\,wu$$

[1] M.C. Fitting, Many-Valued Modal Logic, Fundamenta Informaticae 15: 235–254 (1991).

Many-valued transitivity

Koutras, Nomikos and Peppas:[1]

Definition

Let $a \in \mathbf{A}$ and let $\mathfrak{F} = (W, D, B)$ be an $\mathbf{A}$-frame. Then D and B are said to be *a-transitive* if for all $u, v, w \in W$ we have that

$$a \wedge \mathrm{D}\, wv \wedge \mathrm{D}\, vu \leq \mathrm{D}\, wu \quad \text{and} \quad a \wedge \mathrm{B}\, wv \wedge \mathrm{B}\, vu \leq \mathrm{B}\, wu$$

[1] C.D. Koutras, C. Nomikos and P. Peppas. Canonicity and Completeness Results for Many-Valued Modal Logics, Journal of applied non-classical logic, 12(1): 7 – 41 (2002).

Many-valued transitivity

Frankowski:[1]

Definition

Let f be any filter of **A** let $\mathfrak{F} = (W, D, B)$ be an **A**-frame. Then D and B are said to be *f-transitive* if for all $u, v, w \in W$ we have that

$$\mathrm{D}\,wv \wedge \mathrm{D}\,vu \to \mathrm{D}\,wu \in f \quad \text{and} \quad \mathrm{B}\,wv \wedge \mathrm{B}\,vu \to \mathrm{B}\,wu \in f$$

[1] S. Frankowski. Definable classes of many valued Kripke frames, Bulletin of the Section of Logic 35(1): 27–36 (2006).

Transitive filtrations

For the basic modal language the *largest filtration* is obtained by using the following rule to describe the accessibility relation:

$$R^t[w][v]$$
$$\iff \text{for all formulas } \varphi \text{ if } \Diamond\varphi \in \Sigma \text{ and } \mathfrak{M}, v \Vdash \varphi \vee \Diamond\varphi,$$
$$\text{then } \mathfrak{M}, w \Vdash \Diamond\varphi.$$

Transitive filtrations

$$D_t^{\mathbf{A}}[w][v] = \bigwedge \{V(v, \varphi \vee \Diamond\varphi) \to V(w, \Diamond\varphi) \mid \Diamond\varphi \in \Sigma\}$$

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Proposition

Suppose Σ *is subformula closed and let* $\mathfrak{M}_t^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, \mathrm{D}_t^{(\Sigma,\mathbf{A})}, \mathrm{B}_t^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$ *where* $D_t^{\mathbf{A}}$ *and* $B_t^{\mathbf{A}}$ *are obtained as described above. Then* $\mathfrak{M}_t^{(\Sigma,\mathbf{A})}$ *is a filtration of* $\mathfrak{M}$ *through* Σ *over* $\mathbf{A}$.