Filtrations for many-valued modal logics

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A logic has the finite model property (FMP) if every formula that is not a theorem of the logic can be refuted in a finite model of the logic. Finite model property results can be used to obtain decidability results for logics [6]. One way that the FMP for modal logics has been established in the literature is through filtration constructions [5, 7]. We note that filtration constructions have also been used to prove completeness theorems [8]. A filtration of a model (of a given logic) is obtained by collapsing the model into a finite number of equivalence classes and defining appropriate accessibility relations between these classes. The key is to define relations on the equivalence classes that preserve truth for all formulas in a given subformula-closed set of formulas. In the literature various different 'prescriptions' can be obtained that describe how a filtration on any given model through any given subformula-closed set of formulas should be defined [4]. These include descriptions of how to form the smallest, largest, symmetric, transitive etc. filtrations.

In the research undertaken here we wish to generalize the methods described above to the setting of many-valued modal logics. In [1, 2] Fitting introduced a family of many-valued modal logics over Heyting algebras where both the valuation and the accessibility relations of the associated Kripke models are many-valued. We briefly recall some of the basic definitions regarding the relational semantics of these logics.

The formulas of the *basic many-valued modal language* over a denumerably infinite set of proposition letters Φ are given by the following recursive definition:

$$\varphi := \bot \mid p \mid p \vee q \mid p \wedge q \mid p \to q \mid \Diamond p \mid \Box p.$$

Let $\mathbf{A}$ be a Heyting algebra. A *many-valued (Kripke) frame* over $\mathbf{A}$ ($\mathbf{A}$-frame) for the basic many-valued modal language is a triple $\mathfrak{F} = (W, D, B)$ with a nonempty universe W and many-valued accessibility relations $D : W \times W \to \mathbf{A}$ and $B : W \times W \to \mathbf{A}$. A *many-valued (Kripke) model* over $\mathbf{A}$ ($\mathbf{A}$-model) for the basic many-valued modal language is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is a many-valued frame and $V : W \times \Phi \to \mathbf{A}$ is a many-valued valuation. The valuation can be extended to all formulas. In particular,

$$V(w, \Diamond\varphi) = \bigvee\{D(w,v) \wedge V(v,\varphi) \mid v \in W\} \text{ and}$$
$$V(w, \Box\varphi) = \bigwedge\{B(w,v) \to V(v,\varphi) \mid v \in W\}.$$

Let $a \in \mathbf{A}$, then a formula φ is said to be *a-true* in a model $\mathfrak{M}$ over $\mathbf{A}$ at $w \in W$, denoted by $\mathfrak{M}, W \Vdash_a \varphi$, if $V(w,\varphi) \geq a$.

We define the equivalence relation with respect to a subformula-closed set of formulas, Σ, on a many-valued model $\mathfrak{M}$ over $\mathbf{A}$ as follows: for $w, v \in W$ and $\varphi \in \Sigma$, $w \leftrightsquigarrow^{\mathbf{A}}_{\Sigma} v$ if, and only if, $\mathfrak{M}, w \Vdash_a \varphi \iff \mathfrak{M}, v \Vdash_a \varphi$ for all $a \in \mathbf{A}$. That is, the model is collapsed into equivalence classes such that states in the same equivalence class satisfy all formulas in Σ to the same degree. Let $[w]^{\mathbf{A}}_{\Sigma}$ denote the equivalence class of $w \in W$.

Next we define filtrations for many-valued modal logics.

Definition 1. *Let* $W^{\mathbf{A}}_{\Sigma} = \{[w] \mid w \in W\}$. *Let* $\mathfrak{M}^{(\Sigma,\mathbf{A})}_f = (W^{\mathbf{A}}_{\Sigma}, \mathrm{D}^{(\Sigma,\mathbf{A})}_f, \mathrm{B}^{(\Sigma,\mathbf{A})}_f, V^{\mathbf{A}}_{\Sigma})$ *be any model such that*

(W) $W^{\mathbf{A}}_f = W^{\mathbf{A}}_{\Sigma}$.

(R1) Let $a \in \mathbf{A}$. *If* $\mathrm{D}\, wv \geq a$, *then* $\mathrm{D}^{(\Sigma,\mathbf{A})}_f[w][v] \geq a$.

(R2) Let $a, a' \in \mathbf{A}$. *If* $\mathrm{D}^{(\Sigma,\mathbf{A})}_f[w][v] \geq a$, *then for every* $\Diamond\varphi \in \Sigma$, *if* $\mathfrak{M}, v \Vdash_{a'} \varphi$, *then*
$\mathfrak{M}, w \Vdash_{a \wedge a'} \Diamond\varphi$.

(R3) Let $a \in \mathbf{A}$. *If* $\mathrm{B}\, wv \geq a$, *then* $\mathrm{B}^{(\Sigma,\mathbf{A})}_f[w][v] \geq a$.

(R4) Let $a, a' \in \mathbf{A}$. *If* $\mathrm{B}^{(\Sigma,\mathbf{A})}_f[w][v] \geq a$, *then for every* $\Box\varphi \in \Sigma$, *if* $\mathfrak{M}, w \Vdash_{a'} \Box\varphi$, *then* $\mathfrak{M}, v \Vdash_{a \wedge a'} \varphi$.

(V) $V^{\mathbf{A}}_f([w], p) = V(w, p)$ *for all* $p \in \Sigma$.

Then $\mathfrak{M}^{(\Sigma,\mathbf{A})}_f$ *is called a* filtration of $\mathfrak{M}$ through Σ over $\mathbf{A}$.

We prove the many-valued filtration theorem.

Theorem 2. *Let* $\mathfrak{M}^{(\Sigma,\mathbf{A})}_f = (W^{\mathbf{A}}_{\Sigma}, \mathrm{D}^{(\Sigma,\mathbf{A})}_f, \mathrm{B}^{(\Sigma,\mathbf{A})}_f, V^{\mathbf{A}}_{\Sigma})$ *be a filtration of* $\mathfrak{M}$ *through a subformula closed set* Σ *over* $\mathbf{A}$. *Then, for all formulas* $\varphi \in \Sigma$, *all states* w *in* $\mathfrak{M}$ *and any truth value* $a \neq 0$ *in* $\mathbf{A}$, *we have that*

$$\mathfrak{M}, w \Vdash_a \varphi \iff \mathfrak{M}^{(\Sigma,\mathbf{A})}_f, [w]^{\mathbf{A}}_{\Sigma} \Vdash_a \varphi.$$

We prove that filtrations exist by showing two ways to define many-valued relations on $W_\Sigma^{\mathbf{A}}$ that satisfy the conditions (R1) – (R4). These form the generalizations of the smallest and largest filtrations, respectively. For example, the smallest many-valued filtration of a given model $\mathfrak{M}$ through Σ over $\mathbf{A}$ has relations $D_s^{\mathbf{A}}$ and $B_s^{\mathbf{A}}$ defined by:

$$\begin{aligned} \mathrm{D}_s^{\mathbf{A}}[w]_\Sigma^{\mathbf{A}}[v]_\Sigma^{\mathbf{A}} &= \bigvee\{\mathrm{D}\, w'v' \mid w' \in [w]_\Sigma^{\mathbf{A}}, v' \in [v]_\Sigma^{\mathbf{A}}\}, \text{ and} \\ \mathrm{B}_s^{\mathbf{A}}[w]_\Sigma^{\mathbf{A}}[v]_\Sigma^{\mathbf{A}} &= \bigvee\{\mathrm{B}\, w'v' \mid w' \in [w]_\Sigma^{\mathbf{A}}, v' \in [v]_\Sigma^{\mathbf{A}}\}. \end{aligned}$$

We order many-valued accessibility relations $\mathrm{R}, \mathrm{R}' : W_\Sigma^{\mathbf{A}} \times W_\Sigma^{\mathbf{A}} \to \mathbf{A}$ in the usual way, i.e., $\mathrm{R} \leq \mathrm{R}'$ if, and only if, $\mathrm{R}[w]_\Sigma^{\mathbf{A}}[v]_\Sigma^{\mathbf{A}} \leq^{\mathbf{A}} \mathrm{R}'[w]_\Sigma^{\mathbf{A}}[v]_\Sigma^{\mathbf{A}}$ for all pairs $[w]_\Sigma^{\mathbf{A}}, [v]_\Sigma^{\mathbf{A}} \in W_\Sigma^{\mathbf{A}}$. We show that the accessibility relations $D_s^{\mathbf{A}}$ and $B_s^{\mathbf{A}}$ obtained via the smallest filtration are the least among all accessibility relations obtainable from filtrations of the given model through Σ over $\mathbf{A}$. Similarly, the accessibility relations obtained via the largest filtration are the greatest. These filtrations are therefore aptly named even in this generalized setting.

We show that the basic many-valued modal logic has the FMP if the truth-value space is finite.

Theorem 3. *Let φ be a basic many-valued modal formula. If φ is satisfiable in an $\mathbf{A}$-model where $\mathbf{A}$ is a* finite *Heyting algebra, then φ is satisfiable on a finite model containing at most 2^m states, where m is the product of $|\varphi|$ and $|\mathbf{A}|$.*

Finally we also describe generalizations of the transitive, symmetric and reflexive filtrations. In turn, this enables us to obtain the FMP for some of the many-valued modal logics considered by Frankowski in [3].

References

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