On the lattice of varieties of De Morgan monoids

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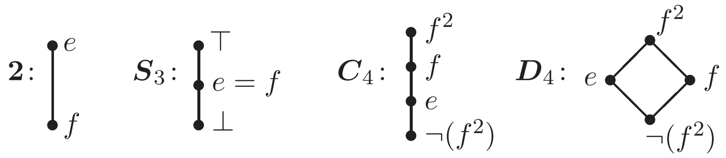
A *De Morgan monoid* $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, e\rangle$ comprises a distributive lattice $\langle A; \wedge, \vee\rangle$, a commutative monoid $\langle A; \cdot, e\rangle$ satisfying $x \leqslant x^2 := x \cdot x$, and a function $\neg\colon A \longrightarrow A$ such that $\boldsymbol{A}$ satisfies $\neg\neg x = x$ and

$$x \cdot y \leqslant z \iff x \cdot \neg z \leqslant \neg y.$$

(The derived operations $x \to y := \neg(x \cdot \neg y)$ and $f := \neg e$ turn $\boldsymbol{A}$ into an involutive residuated lattice in the sense of [4].) The class DM of all De Morgan monoids is a variety that algebraizes the relevance logic $\mathbf{R^t}$ of [1]. Its lattice of subvarieties Λ_{DM} is dually isomorphic to the lattice of axiomatic extensions of $\mathbf{R^t}$. We study the former here, as a route to the latter.

To describe the atoms of Λ_{DM}, we need to refer to the De Morgan monoids depicted below. Note that, if b is the least element of a De Morgan monoid, then $a \cdot b = b$ for all elements a. In what follows, $\mathbb{V}(\boldsymbol{A})$ denotes the smallest variety containing an algebra $\boldsymbol{A}$.



Lemma 1. *Up to isomorphism,* $\mathbf{2}$, $\boldsymbol{C}_4$ *and* $\boldsymbol{D}_4$ *are the only simple* 0*–generated De Morgan monoids.*

Theorem 2. *The distinct classes* $\mathbb{V}(\mathbf{2})$, $\mathbb{V}(\boldsymbol{S}_3)$, $\mathbb{V}(\boldsymbol{C}_4)$ *and* $\mathbb{V}(\boldsymbol{D}_4)$ *are precisely the minimal (nontrivial) varieties of De Morgan monoids.*

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Lemma 1 is implicit in Slaney's identification of the 0–generated finitely subdirectly irreducible De Morgan monoids [10], but it is easier to prove it directly. Theorem 2 (which uses Lemma 1 and universal algebraic methods) does not seem to have been stated explicitly in the relevance logic literature.

To understand Λ_{DM}, it helps to know, for a given variety K of De Morgan monoids, whether every proper subquasivariety of K generates a proper subvariety of K. If this is true, then K is said to be *structurally complete*, and so is the axiomatic extension $\mathbf{L}$ of $\mathbf{R^t}$ corresponding to K. This means that every proper extension of $\mathbf{L}$ (really of $\vdash_{\mathbf{L}}$) has some new theorem—as opposed to having nothing but new rules of derivation, cf. [2].

It has recently been shown that the relevance logic $\mathbf{R}$ of [1] (which is $\mathbf{R^t}$ without the Ackermann constants $\mathbf{t}$ and $\mathbf{f}$) has no structurally complete axiomatic consistent extension, except for classical propositional logic [9]. In fact, no such extension of $\mathbf{R}$ is even *passively* structurally complete (PSC). In general, a logic is PSC if each of its underivable finite rules $\alpha_1, \ldots, \alpha_n \,/\, \alpha$ $(n \geq 1)$ has the property that some substitution turns all of $\alpha_1, \ldots, \alpha_n$ into theorems. A variety K algebraizing a PSC logic is also said to be PSC, and this means that every existential positive sentence in the first order language of K holds either in all or in none of the nontrivial members of K, cf. [11]. Note that subvarieties of PSC varieties are PSC, in contrast with the case of structural completeness.

The analysis of these notions in $\mathbf{R^t}$ turns out to be considerably more complex than in the case of $\mathbf{R}$. We are a long way from knowing which elements of Λ_{DM} are structurally complete, but the following result settles the case of PSC.

Theorem 3. *A variety K of De Morgan monoids is passively structurally complete iff one of the following four (mutually exclusive) conditions holds*:

(i) K *is the variety* $\mathbb{V}(\mathbf{2})$ *of all Boolean algebras;*

(ii) $\mathsf{K} = \mathbb{V}(\boldsymbol{D}_4)$;

(iii) K *consists of odd Sugihara monoids;*

(iv) $\boldsymbol{C}_4$ *is a retract of every nontrivial member of* K.

Here, an *odd Sugihara monoid* is a De Morgan monoid in which $f = e$ (whence, as it happens, $a^2 = a$ for all elements a). Algebras of this kind are very well understood, cf. [6, 7]. In particular, the lattice of varieties of odd Sugihara monoids is a chain of order type $\omega + 1$, with $\mathbb{V}(\boldsymbol{S}_3)$ as its atom. The De Morgan monoids that have $\boldsymbol{C}_4$ as a retract (or are trivial) form a quasivariety that is not a variety, so the following result is not obvious:

Theorem 4. *The largest variety* M *of De Morgan monoids satisfying the demand in Theorem 3(iv) exists, and it is finitely axiomatized.*

The varieties described in items (i)–(iii) of Theorem 3 are in fact structurally complete, as is $\mathbb{V}(\boldsymbol{C}_4)$, but the situation for the other subvarieties of M is more complicated. A construction in [5, Sec. 9] (with antecedents in [8]) embeds any Brouwerian algebra into a De Morgan monoid belonging to M. It induces an injective function from varieties of Brouwerian algebras to subvarieties of M that preserves structural *incompleteness*. It is well known that the variety of all Brouwerian algebras is not structurally complete, so the same applies to at least some subvarieties of M.

These remarks suggest that the classification of structurally complete varieties of De Morgan monoids might reduce, at least partially, to the corresponding problem for Brouwerian algebras. The latter problem does not appear to have been solved. The nearest result seems to be Citkin's classification of the *hereditarily* structurally complete (HSC) varieties of Heyting algebras. (A variety is HSC iff all of its subquasivarieties are varieties.) Not every structurally complete variety of Heyting algebras is HSC; see [3].

References

[1] A.R. Anderson, N.D. Belnap, Jnr., 'Entailment: The Logic of Relevance and Necessity, Volume 1', Princeton University Press, 1975.

[2] C. Bergman, *Structural completeness in algebra and logic*, in H. Andréka, J.D. Monk and I. Nemeti (eds.), 'Algebraic Logic', Colloquia Mathematica Societatis János Bolyai Vol. 54, North-Holland, Amsterdam, 1991, pp. 59–73.

[3] A.I. Citkin, *On structurally complete superintuitionistic logics* (Russian), Dokl. Akad. Nauk SSSR **241** (1978), 40–43. English translation in Soviet Math. Dokl. **19** (1978), 816–819.

[4] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, 'Residuated Lattices. An Algebraic Glimpse at Substructural Logics', Elsevier, 2007.

[5] N. Galatos, J.G. Raftery, *Adding involution to residuated structures*, Studia Logica **77** (2004), 181–207.

[6] N. Galatos, J.G. Raftery, *A category equivalence for odd Sugihara monoids and its applications*, J. Pure Appl. Algebra **216** (2012), 2177–2192.

[7] J.S. Olson, J.G. Raftery, *Positive Sugihara monoids*, Algebra Universalis **57** (2007), 75–99.

[8] R.K. Meyer, *On conserving positive logics*, Notre Dame J. Formal Logic **14** (1973), 224–236.

[9] J.G. Raftery, K. Świrydowicz, *Structural completeness in relevance logics*, Studia Logica, to appear.

[10] J.K. Slaney, *3088 varieties: a solution to the Ackermann constant problem*, J. Symbolic Logic **50** (1985), 487–501.

[11] A. Wroński, *Overflow rules and a weakening of structural completeness*, in J. Sytnik-Czetwertyński (ed.), 'Rozważania o Filozofii Prawdziwej. Jerzemu Perzanowskiemu w Darze', Wydawnictwo Uniwersytetu Jagiellońskiego, Kraków, 2009, pp. 67–71.