# Epimorphism surjectivity and the Beth definability property

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- Are epis surjective in a variety?
- Yes: Boolean algebras, Heyting algebras, lattices, semilattices and (Abelian) groups.
- No: distributive lattices, rings with unity and monoids.
- Thus epimorphism surjectivity is not preserved in subvarieties!

# Beth property

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- $\mathcal{L}$ has the (resp. finite) Beth property when 1 (resp. with $Z$ finite) implies 2.

# Transfer theorem

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$$\text{Beth property} \neq \text{finite Beth property.}$$

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**Theorem**

In varieties of Heyting algebras almost onto epis are surjective.

- To establish Blok and Hoogland's conjecture, it is enough to find a variety of Heyting algebras where epis need not be surjective.

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$$\mathsf{K} \vDash \forall \vec{x}, y, z((\varphi(\vec{x}, y) \,\&\, \varphi(\vec{x}, z)) \to y \approx z) \text{ and } \boldsymbol{B} \vDash \varphi(\vec{a}, b).$$

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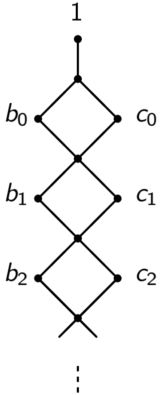

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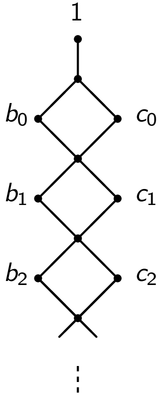


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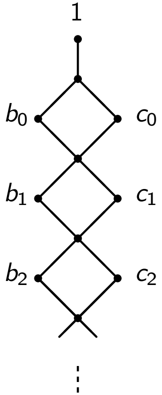


- We claim that $\boldsymbol{B}$ is a $\mathbb{V}(\boldsymbol{A})$-epic subalgebra of $\boldsymbol{A}$.
- We need to find primitive positive formulas that define partial functions in $\mathbb{V}(\boldsymbol{A})$ and, moreover, construct $\boldsymbol{A}$ out of $\boldsymbol{B}$.

# Partial functions

- Consider the conjunction of equations

$$\varphi(x_0, x_1, x_2, y_0, y_1, y_2) \coloneqq \mathop{\&}_{n \leq 2} (x_n \to y_n \approx y_n \,\&\, y_n \to x_n \approx x_n)$$
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- $\Phi$ define a partial 3-ary function in $\mathbb{V}(\boldsymbol{A})$: For every $\boldsymbol{C} \in \mathbb{V}(\boldsymbol{A})$ and $a_0, a_1, a_2 \in C$ there is at most one $e \in C$ s.t.

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$$\boldsymbol{A} \vDash \Phi(b_{n+2}, b_{n+1}, b_n, c_n) \text{ for every } n \in \omega.$$

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**Theorem (Blok-Hoogland's conjecture)**

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2. The Beth property and the finite Beth property are different in locally tabular superintuitionistic logics.

# Rieger-Nishimura lattice

**Definition**

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2. Among them there is the variety generated by the Rieger-Nishimura lattice.

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**Theorem**

Let K be a variety of Heyting algberas. If K has finite depth, then epimorphisms are surjective in K.

# Consequences

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## Corollary

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**Corollary**

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2. Tabular superintuitionistic logics have the Beth property.
3. Superintuitionistic logics, whose theorems include $h_n$ for some $n \in \omega$, have the Beth property.
4. Epimorphisms are surjective in all varieties of Gödel algebras.

# Strong epimorphism surjectivity

## Definition

A class of algebras K has strong epimorphism surjectivity if whenever $f\colon \boldsymbol{A} \to \boldsymbol{B}$ is homomorphism in K and $b \in B \smallsetminus f(A)$, there are homomorphisms $g, h\colon \boldsymbol{B} \rightrightarrows \boldsymbol{C}$ in K such that

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### Theorem (Maksimova)

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- There is a continuum of varieties of depth $\leq 3$.
- Thus there is a continuum of varieties with epimorphism surjectivity but not strong epimorphism surjectivity.

Thanks for coming!