

A Modal Logic of the Real Numbers

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An Axiomatization Problem

Hansoul and Teheux (2013) axiomatize a **modal Łukasiewicz logic** over (crisp) Kripke frames by adding to an axiomatization of Łukasiewicz logic

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$

$$\Box(\varphi \oplus \varphi) \rightarrow (\Box\varphi \oplus \Box\varphi)$$

$$\Box(\varphi \odot \varphi) \rightarrow (\Box\varphi \odot \Box\varphi)$$

$$\frac{\varphi}{\Box\varphi}$$

and a rule with infinitely many premises

$$\frac{\varphi \oplus \varphi \quad \varphi \oplus \varphi^2 \quad \varphi \oplus \varphi^3 \quad \dots}{\varphi}$$

But is this infinitary rule really necessary?

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But is this infinitary rule really necessary?

We axiomatize a **modal logic of the real numbers** that extends the multiplicative fragment of Abelian logic.

D. Diaconescu, G. Metcalfe, and L. Schnüriger.
Axiomatizing a Real-Valued Modal Logic.
Proceedings of AiML 2016, to appear.

The Multiplicative Fragment of Abelian Logic

The **multiplicative fragment of abelian logic** is axiomatized by

$$(B) \quad (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$$

$$(C) \quad (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$$

$$(I) \quad \varphi \rightarrow \varphi$$

$$(A) \quad ((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow \varphi$$

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} \text{ (mp)}$$

and is complete with respect to the logical matrix

$$\langle (\mathbb{R}, \rightarrow^{\mathbb{R}}), \mathbb{R}_{\geq 0} \rangle \quad \text{where } x \rightarrow^{\mathbb{R}} y = y - x.$$

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We define further connectives (for a fixed variable p_0)

$$\bar{0} := p_0 \rightarrow p_0$$

$$\neg\varphi := \varphi \rightarrow \bar{0}$$

$$\varphi + \psi := \neg\varphi \rightarrow \psi.$$

For our modal language, we add a unary connective \Box , and define

$$\Diamond\varphi := \neg\Box\neg\varphi.$$

The set of formulas \mathbb{F}_m for this language is defined inductively as usual over a countably infinite set of variables $\mathbb{V}\text{ar}$.

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A **(crisp) frame** $\mathfrak{F} = \langle W, R \rangle$ consists of

- a non-empty set of **worlds** W
- an **accessibility relation** $R \subseteq W \times W$.

\mathfrak{F} is called **serial** if for all $x \in W$, there exists $y \in W$ such that Rxy .

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A $K(\mathbb{R})$ -**model** $\langle W, R, V \rangle$ consists of

- a **serial frame** $\langle W, R \rangle$
- an **evaluation map** $V: \text{Var} \times W \rightarrow D$ for some bounded $D \subseteq \mathbb{R}$.

The evaluation map is extended to $V: \text{Fm} \times W \rightarrow \mathbb{R}$ by

$$V(\varphi \rightarrow \psi, x) = V(\psi, x) - V(\varphi, x)$$

$$V(\Box\varphi, x) = \inf\{V(\varphi, y) : Rxy\}.$$

It follows also that

$$V(\bar{0}, x) = 0$$

$$V(\varphi + \psi, x) = V(\varphi, x) + V(\psi, x)$$

$$V(\neg\varphi, x) = -V(\varphi, x)$$

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A formula φ is

- **valid** in a $K(\mathbb{R})$ -model $\langle W, R, V \rangle$ if $V(\varphi, x) \geq 0$ for all $x \in W$
- $K(\mathbb{R})$ -**valid** if it is valid in all $K(\mathbb{R})$ -models.

Lemma

The following are equivalent for any formula φ :

- (1) φ is $K(\mathbb{R})$ -valid.
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An Axiom System

Our axiom system $K(\mathbb{R})$ consists of an axiomatization for the multiplicative fragment of abelian logic extended with

$$(K) \quad \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$

$$(P) \quad \Box(\varphi + \dots + \varphi) \rightarrow (\Box\varphi + \dots + \Box\varphi)$$

$$\frac{\varphi}{\Box\varphi} \text{ (nec)} \qquad \frac{\varphi + \dots + \varphi}{\varphi} \text{ (con)}$$

The Sequent Calculus $GK(\mathbb{R})$

$$\frac{}{\Delta \Rightarrow \Delta} \text{ (ID)}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Pi \Rightarrow \varphi, \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta} \text{ (CUT)}$$

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta} \text{ (MIX)}$$

$$\frac{\Gamma, \dots, \Gamma \Rightarrow \Delta, \dots, \Delta}{\Gamma \Rightarrow \Delta} \text{ (SC)}$$

$$\frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta} \text{ (}\rightarrow\Rightarrow\text{)}$$

$$\frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \rightarrow \psi, \Delta} \text{ (}\Rightarrow\rightarrow\text{)}$$

$$\frac{\Gamma \Rightarrow \varphi, \dots, \varphi}{\Box \Gamma \Rightarrow \Box \varphi, \dots, \Box \varphi} \text{ (}\Box\text{)}$$

Equivalence of Proof Systems

We interpret sequents by

$$(\varphi_1, \dots, \varphi_n \Rightarrow \psi_1, \dots, \psi_m)^{\mathcal{I}} := (\varphi_1 + \dots + \varphi_n) \rightarrow (\psi_1 + \dots + \psi_m),$$

where $\varphi_1 + \dots + \varphi_n := \bar{0}$ for $n = 0$.

Theorem

The following are equivalent:

- (1) $\Gamma \Rightarrow \Delta$ is derivable in $\text{GK}(\mathbb{R})$.
- (2) $(\Gamma \Rightarrow \Delta)^{\mathcal{I}}$ is derivable in $\text{K}(\mathbb{R})$.

Moreover, $\text{GK}(\mathbb{R})$ admits cut elimination.

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Theorem

The following are equivalent for any formula φ :

- (1) φ is $\mathbf{K}(\mathbb{R})$ -valid.*
- (2) φ is derivable in $\mathbf{K}(\mathbb{R})$.*
- (3) $\Rightarrow \varphi$ is derivable in $\mathbf{GK}(\mathbb{R})$.*

We prove by induction on the complexity of a sequent S that

$$S^{\mathcal{I}} \text{ is } \mathbb{K}(\mathbb{R})\text{-valid} \implies S \text{ is derivable in } \text{GK}(\mathbb{R}).$$

Suppose that S is $\Box\Gamma \Rightarrow \Box\varphi_1, \dots, \Box\varphi_n$. We apply the following $\text{GK}(\mathbb{R})$ -derivable rule for some $k > 0$ and $k\Gamma = \Gamma_0, \Gamma_1, \dots, \Gamma_n$:

$$\frac{\Gamma_0 \Rightarrow \Gamma_1 \Rightarrow k[\varphi_1] \quad \dots \quad \Gamma_n \Rightarrow k[\varphi_n]}{\Box\Gamma \Rightarrow \Box\varphi_1, \dots, \Box\varphi_n}$$

Using the $\mathbb{K}(\mathbb{R})$ -validity of S , we generate (via labelled tableau rules) an inconsistent set of linear inequations. This inconsistency is witnessed by a linear combination of sequents where k is the coefficient of S . Eliminating variables we get the $\mathbb{K}(\mathbb{R})$ -validity of $\Gamma_0 \Rightarrow, \Gamma_1 \Rightarrow k[\varphi_1], \dots, \Gamma_n \Rightarrow k[\varphi_n]$.

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Using our labelled tableau rules, we also obtain:

Theorem

Checking $K(\mathbb{R})$ -validity of formulas is in EXPTIME.

There remain many issues to resolve:

- Can we add extend our axiomatization to an “Abelian modal logic” with lattice connectives? Can we obtain Łukasiewicz modal logic?
- Can we develop useful algebraic semantics for these logics?
- Is the complexity of $\mathbb{K}(\mathbb{R})$ -validity EXPTIME-complete? What is the complexity of validity in Łukasiewicz modal logic?

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