Correspondence Theory for Many-Valued Modal Logics

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Outline of our basic MVML $L_{\mathbb{A}}$

- We use the framework suggested by Fitting.
- Truth-value space: A perfect Heyting algebra $\mathbb{A}$.
- Formulas of $L_{\mathbb{A}}$:

$$\theta := \mathbf{t} \mid p \mid \theta \vee \psi \mid \theta \wedge \psi \mid \theta \rightarrow \psi \mid \Box\theta \mid \Diamond\theta$$

- Given $\mathfrak{F} = (W, D, B)$ (crisp or non-crisp), where $V : (Prop \times W) \rightarrow A$.

Example

- $V(\mathbf{t}, w) = t \in A$
- $V(\Diamond p, w) = \bigvee\{V(p, u) \mid Rwu\}$
- $V(\Box p, w) = \bigwedge\{V(p, u) \mid Rwu\}$

a-Validity

a-Validity

Let $a \in A$. An $L_{\mathbb{A}}$ formula ϕ is *a-valid* in a frame $\mathfrak{F}$ at $w \in W$ ($\mathfrak{F}, w \Vdash_a \phi$) if $V(\phi, w) \geq a$ for all valuations V on $\mathfrak{F}$.

Example

$$\mathbb{A} = (\{0, \tfrac{1}{2}, 1\}, \wedge, \vee, 0, 1)$$

1
|
$\mathbf{\frac{1}{2}}$
|
0

Let $\mathfrak{F} = (W, D, B)$ be crisp.

1. Classically: $\mathfrak{F}, w \Vdash p \vee \neg p$.
2. MVML: $\mathfrak{F}, w \nVdash_1 p \vee \neg p$ (Let $V(p, w) = \frac{1}{2}$)

Outline of our extended MVML $L_{\mathbb{A}}^{+}$

- ALBA is an algorithm that computes FO frame correspondents for certain modal formulas.
- For ALBA we need some extra machinery.
- $L_{\mathbb{A}}^{+}$ is $L_{\mathbb{A}}$ with extra formulas $\mathbf{i}, \mathbf{m}, \theta - \psi, \blacklozenge\theta$ and $\blacksquare\theta$.

The valuation V

Given a valuation V for $L_{\mathbb{A}}^{+}$:

1. $V(\mathbf{i}, w_0) \in J^{\infty}(\mathbb{A})$ for exactly one $w_0 \in W$ and $V(\mathbf{i}, w) = 0$ whenever $w_0 \neq w$.
2. $V(\mathbf{m}, w_1) \in M^{\infty}(\mathbb{A})$ for exactly one $w_1 \in W$ and $V(\mathbf{m}, w) = 1$ whenever $w_1 \neq w$.

Outline of our basic MVFOL $L_{\mathbb{A}}^{FO}$

- Truth-value space: Perfect Heyting algebra $\mathbb{A}$.
- Unary many-valued predicates P_i corresponding to $p_i \in Prop$.
- Equality and $\leq$
- Binary relation symbols D and B.
- Assignments: $v : FORM_{\mathbb{A}} \to A$.

a-Truth

Let $a \in A$. An $L_{\mathbb{A}}^{FO+}$ formula β is *a-true* in an interpretation I ($I \vDash_a \beta$) if $v(\beta) \geq a$ for all assignments v in I.

Outline of our extended MVFOL $L_{\mathbb{A}}^{FO+}$

- $L_{\mathbb{A}}^{FO+}$ is $L_{\mathbb{A}}^{FO}$ with extra formulas $A - B$, constant symbols $c_{\mathbf{i}}$ and $c_{\mathbf{m}}$ and truth constants $C_{\mathbf{i}}$ and $C_{\mathbf{m}}$.
- Informally, an assignment will map $c_{\mathbf{i}}$ to a point and $C_{\mathbf{i}}$ to a truth value.

Correspondence

Informally, in terms of correspondence: Let $\mathfrak{F} = (W, D, B)$, then

1. $v(c_{\mathbf{i}}) = w_0$
2. $v(C_{\mathbf{i}}) = V(\mathbf{i}, w_0) \in J^{\infty}(\mathbb{A})$

The Standard Translation for $L_{\mathbb{A}}^{+}$

Standard Translation

1. $ST_x(\mathbf{t}) = \mathbf{t}$
2. $ST_x(\mathbf{i}) = (c_{\mathbf{i}} = x) \wedge C_{\mathbf{i}}$
3. $ST_x(\mathbf{m}) = (c_{\mathbf{m}} \neq x) \vee C_{\mathbf{m}}$

Example

$$ST_x((\Diamond p_n \vee \mathbf{i}) \to \mathbf{t}) \quad = \quad (\exists y(Rxy \wedge P_n(x)) \vee ((x = c_i) \wedge C_i)) \to \mathbf{t}$$

Proposition

Let $\mathfrak{M} = (\mathfrak{F}, V)$ be an $\mathbb{A}$-model, $a \in A$, $w \in W$ and ϕ a formula of $L_{\mathbb{A}}^{+}$. Let x be a free variable of $L_{\mathbb{A}}^{FO+}$ and v any assignment on the first-order interpretation $\mathfrak{M}^{FO+}$ such that $v(x) = w$. Then:

$$V(\phi, w) = v(ST_x(\phi))$$

Proposition

Let ϕ be a formula $L_{\mathbb{A}}^{+}$. Let x be a free variable of $L_{\mathbb{A}}^{FO+}$. Then:

$$\mathfrak{F}, w \Vdash_a \phi \quad \text{iff} \quad \mathfrak{F} \vDash_a \forall\overline{P}\forall\overline{\mathbf{c_i}}\forall\overline{\mathbf{c_m}}\forall\overline{\mathbf{C_i}}\forall\overline{\mathbf{C_m}}(ST_x(\phi)[x := w]$$

Correspondence

- Well-developed in the Classical setting, e.g. the Sahlqvist class.
- ALBA is an algorithm that computes FO frame correspondents for certain modal formulas.
- Aim: Extend ALBA to the many-valued setting.

Local *a*-Frame Correspondent

Let $\phi \in L_{\mathbb{A}}^{+}$, $\alpha \in L_{\mathbb{A}}^{FO+}$ and $a \in A$. ϕ and α are local frame a-correspondents if

$$\mathfrak{F}, w \Vdash_a \phi \quad \text{iff} \quad \mathfrak{F} \vDash_a \alpha[x := w]$$

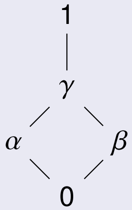
Examples

Example

Fix $\mathbb{A}_1 = (\{0, \frac{1}{2}, 1\}, \wedge, \vee, \rightarrow, 0, 1)$.

- $p \rightarrow \Diamond p$ and Rxx are local frame a-correspondents, $a \in \{\frac{1}{2}, 1\}$.

Example



- $\neg p \vee \Diamond p$ and 0 are local frame 1-correspondents.
- $\neg p \vee \Diamond p$ and Rxx are local frame γ-correspondents.

$\mathbb{A}$-MAs

$\mathbb{A}$-MAs

Fix $W \neq \emptyset$. An $\mathbb{A}$-MA is a power algebra $\mathbb{G} = (\mathbb{A}^W, \Diamond, \Box, \{\mathbf{t}\}_{t \in A})$ s.t. $\forall x \in W$ and $\forall f \in \mathbb{A}^W$:

$$(\Diamond f)(x) \leq \bigvee \left\{ f(y) \wedge \bigwedge \left\{ g(y) \rightarrow (\Diamond g)(x) \mid g \in A^W \right\} \mid y \in W \right\}$$

$$(\Box f)(x) \geq \bigwedge \left\{ \bigwedge \left\{ (\Box g)(x) \rightarrow g(y) \mid g \in A^W \right\} \rightarrow f(y) \mid y \in W \right\}$$

a-validity

Given $a \in A$, we say that $\phi \leq \psi$ is a-valid in $\mathbb{G}$ ($\mathbb{G} \models_a \phi \leq \psi$) if under all v on $\mathbb{G}$, we have that $\phi \wedge \mathbf{a} \leq \psi$.

Complex Algebras and Prime Structures

- If $\mathfrak{F} = (W, D, B)$, then $\mathfrak{F}^+ = (\mathbb{A}^W, \Diamond_D, \Box_B, \{\mathbf{t}\}_{t\in A})$ s.t.

$$(\Diamond_D f)(x) = \bigvee \left\{ f(y) \wedge D(x,y) \mid y \in W \right\}$$
$$(\Box_B f)(x) = \bigwedge \left\{ B(x,y) \to f(y) \mid y \in W \right\}$$

- If $\mathbb{G} = (\mathbb{A}^W, \Diamond, \Box, \{\mathbf{t}\}_{t\in A})$, then $\mathbb{G}_\circ = (W, D_\Diamond, B_\Box)$ s.t.

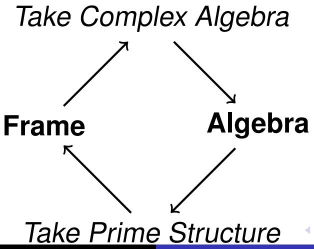
$$D_\Diamond(x,y) = \bigwedge \left\{ g(y) \to (\Diamond g)(x) \mid g \in A^W \right\}$$
$$B_\Box(x,y) = \bigwedge \left\{ (\Box g)(x) \to g(y) \mid g \in A^W \right\}$$

Duality

Duality Result

Let $\mathfrak{F} = (W, D, B)$ be an $\mathbb{A}$-frame and let $\mathbb{G} = (\mathbb{A}^W, \Diamond, \Box, \{\boldsymbol{t}\}_{t \in A})$ be an $\mathbb{A}$-MA. Then:

1. $\mathbb{G} \cong (\mathbb{G}_\circ)^+$
2. $\mathfrak{F} \cong (\mathfrak{F}^+)_\circ$



ALBA vs. MV-ALBA

ALBA vs.

1. $\phi \leq \psi$
2. First Approx.: $\mathbf{i_0} \leq \phi$; $\psi \leq \mathbf{m_0}$
3. Initial System: $\{\mathbf{i_0} \leq \phi; \psi \leq \mathbf{m_0}\}$
4. Apply ALBA

MV-ALBA

1. $\phi \wedge \mathbf{a} \leq \psi$
2. First Approx.: $\mathbf{i_0} \leq \phi \wedge \mathbf{a}$; $\psi \leq \mathbf{m_0}$
3. Splitting: $\mathbf{i_0} \leq \phi$; $\mathbf{i_0} \leq \mathbf{a}$; $\psi \leq \mathbf{m_0}$
4. Initial System: $\{\mathbf{i_0} \leq \mathbf{a}, \mathbf{i_0} \leq \phi; \psi \leq \mathbf{m_0}\}$
5. Apply ALBA

Theorem

Theorem

ALBA can be applied to both the inductive- and the Sahlqvist classes of many-valued formulas of $L^{+}_{\mathbb{A}}$.

- The inductive formulas extend the Sahlqvist class.

Example

Example

Input: $p \wedge \mathbf{a} \leq \Diamond p$

- First Approximation: $(\mathbf{i_0} \leq p \wedge \mathbf{a} \ \&\ \Diamond p \leq \mathbf{m_0}) \Rightarrow (\mathbf{i_0} \leq \mathbf{m_0})$
- Splitting: $(\mathbf{i_0} \leq \mathbf{a} \ \&\ \mathbf{i_0} \leq p \ \&\ \Diamond p \leq \mathbf{m_0}) \Rightarrow (\mathbf{i_0} \leq \mathbf{m_0})$
- Right Ackermann Rule: $(\mathbf{i_0} \leq \mathbf{a} \ \&\ \Diamond \mathbf{i_0} \leq \mathbf{m_0}) \Rightarrow (\mathbf{i_0} \leq \mathbf{m_0})$
- Output: $$\begin{aligned} &= && \forall c_{m_0} \forall C_{m_0} \forall C_{i_0} [\forall x(((x = c_{i_0}) \wedge C_{i_0} \leq a) \\ &\wedge && \exists y(Rxy \wedge (c_{i_0} = y) \wedge C_{i_0}) \\ &\leq && ((x \neq c_{m_0}) \vee C_{m_0}))] \\ &\Rightarrow && \forall x((c_{i_0} = x) \wedge C_{i_0} \leq (c_{m_0} \neq x) \vee C_{m_0}) \end{aligned}$$
- Output simplifies to $Rc_{i_0}c_{i_0}$.

The Sahlqvist Class

- In two-valued case we have that:

Theorem

Let $\phi \to \psi$ be a Sahlqvist formula in the basic modal language L. Then:

$$\mathfrak{F}, w \Vdash \phi \to \psi \quad \text{iff} \quad \mathfrak{F} \models \alpha[x := w]$$

Moreover, α is effectively computable from $\phi \to \psi$.

- Question: Given a 2-valued Sahlqvist formula and its classical local frame correspondent, are they also local frame a-correspondents?

Restricted Sahlqvist Class

Restricted Very Simple Sahlqvist Formula

A *restricted very simple Sahlqvist formula* is an implication $\phi \to \psi$ in which ϕ is a very simple Sahlqvist antecedent and ψ satisfies:

1. ψ is positive.
2. For each $p \in Prop$ in ψ, p does not occur in any subformula α such that $\alpha \to \gamma$ is a subformula of ϕ.

Theorem

Let $\phi \to \psi$ be a restricted very simple Sahlqvist formula and let α be its classical local frame correspondent. Then:

$$\mathfrak{F}, w \Vdash_a \phi \to \psi \quad \text{iff} \quad \mathfrak{F} \vDash_a \alpha[x := w]$$

Work to be completed

- Extend *a*-correspondence results to restricted simple Sahlqvist and restricted Sahlqvist formulas (for crisp and non-crisp cases).
- Investigate dependence of correspondence on choice of truth-value (e.g. $\neg p \vee \Diamond p$).

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