A new hierarchy of infinitary propositional logics

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Most of the literature on non-classical propositional logics is devoted to systems that, just like classical logic, are *finitary*, in the sense that whenever a proposition follows from a set of premises, it must also follow from a *finite* subset of these premises. Such restriction is due to the fact that finitarity is a technically convenient assumption that substantially simplifies the necessary mathematical framework. Moreover, it may also be argued, from a more philosophical point of view, that if mathematical logic is supposed to model *correct reasoning*, then it should provide systems that, like a finite rational being, can only perform finitely-many inference steps to justify a proposition. However, beyond that motivation, one can as well find many natural examples of *infinitary* logics in the literature, i.e. systems where a proposition may follow from an infinite set of premises, but not from any of its finite subsets, or equivalently, systems that need infinitary inference when presented in terms of a proof calculus. A prominent one is the infinitely-valued Łukasiewicz logic $Ł_\infty$. Therefore, abstract algebraic logic, a discipline that intends to provide a very general and encompassing approach to the study of non-classical logics, cannot be restricted only to finitary logics.

In this talk we focus on the completeness properties of (in)finitary logics, in particular we generalize the well-known result asserting that any finitary logic is complete with respect to its relatively (finitely) subdirectly irreducible models. To this end, we describe two properties of the closure system of theories which are naturally fulfilled by all finitary logics and imply the mentioned completeness properties.

More precisely, let L be a propositional logic (i.e. a structural consequence relation) and let Th(L) be the closure system all of its theories. The class of (reduced) matrix models of a logic L is denoted as $\mathbf{MOD}(\mathrm{L})$ (or $\mathbf{MOD}^*(\mathrm{L})$ respectively). Both classes give complete semantics for any logic L; however it is common to consider meaningful subclasses of reduced models which may provide stronger completeness theorems. A matrix $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ is *relatively (finitely) subdirectly irreducible in* $\mathbf{MOD}^*(\mathrm{L})$, in symbols $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ ($\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$), if it cannot be decomposed as a non-trivial subdirect product of an arbitrary (finite non-empty) family of matrices from

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$\mathbf{MOD}^*(\mathrm{L})$. We say L has the *intersection prime extension property*, IPEP, provided the system $\{T \in \mathrm{Th(L)} \mid T$ is finitely $\cap$-irreducible$\}$ forms a base of Th(L) (i.e. any consistent theory can be expressed as an intersection of finitely $\cap$-irreducible theories). The property IPEP was first described in [2]. The notion of finitely $\cap$-irreducible theory generalizes concepts of linear theories (for any pair of formulae φ, ψ either $\varphi \to \psi \in T$ or $\psi \to \varphi \in T$) which play an important role in study of fuzzy logics and, in case of logics with disjunction, it coincides with the important notion of prime theory ($\varphi \vee \psi \in T$ then either $\varphi \in T$ or $\psi \in T$) (see [1, 2, 3]). Further we define a natural strengthening of the IPEP, namely the *completely intersection prime extension property*, CIPEP, requiring the set $\{T \in \mathrm{Th(L)} \mid T$ is $\cap$-irreducible$\}$ to be a base of Th(L) . Further we say L is R(F)SI-*complete* if $\mathrm{L} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{R(F)SI}}}$, i.e. if the logic is complete w.r.t. its relatively (finitely) subdirectly irreducible models. The stipulated relations among these properties are depicted on Figure 1.

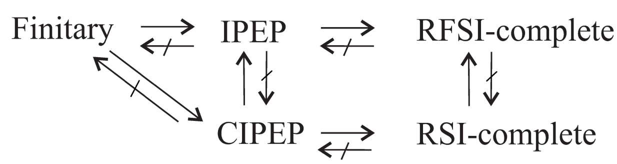

Figure 1: The new hierarchy

The implications depicted by crossed-barred arrows are trivial or well-known. The difficult part is finding the necessary counterexamples to show the crossed arrows. The classes are separated by producing three examples of *infinitary* logics:

1. a logic with the CIPEP,

2. a logic with the IPEP which is not RSI-complete, and

3. an RSI-complete logic without the IPEP.

A nice fact is, that all the mentioned examples can be found in the realm of protoalgebraic logics (the first two even at the top of the Leibniz hierarchy). In order to obtain examples 1. and 2. we present a semantical characterization for the IPEP and CIPEP in case of protoalgebraic logics using a very helpful notion of surjective semantical consequence, capitalizing on the correspondence theorem for protoalgebraic logics (see [4]). All these results are presented in the submitted paper [5].

References

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