A new hierarchy of infinitary propositional logics

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Outline

1. Introduction
2. Motivation
3. New hierarchy

Basic notions of AAL- syntax

Language $\mathcal{L}$ (set of connectives)

Set of variables Var (usually countable) $\ldots x, y, z$

Algebra of formulae $\boldsymbol{Fm}_{\mathcal{L}}$ with underlying set $Fm_{\mathcal{L}}$
formulae $\ldots \varphi, \psi, \chi,$
sets of formulae$\ldots \Gamma, \Delta$

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Logic L is a *structural closure operator* Th_{L} on $Fm_{\mathcal{L}}$, i.e.
$\mathrm{Th}_{\mathrm{L}} : P(Fm_{\mathcal{L}}) \to P(Fm_{\mathcal{L}})$

1. $\Gamma \subseteq \mathrm{Th}_{\mathrm{L}}(\Gamma)$ (*extensivity*)
2. $\Gamma \subseteq \Delta$ then $\mathrm{Th}_{\mathrm{L}}(\Gamma) \subseteq \mathrm{Th}_{\mathrm{L}}(\Delta)$ (*monotonicity*)
3. $\mathrm{Th}_{\mathrm{L}}(\mathrm{Th}_{\mathrm{L}}(\Gamma)) = \mathrm{Th}_{\mathrm{L}}(\Gamma)$ (*idempotency*)
4. $\sigma\mathrm{Th}_{\mathrm{L}}(\Gamma) \subseteq \mathrm{Th}_{\mathrm{L}}(\sigma\Gamma)$, where σ is a substitution (*structurality*)

We write $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\varphi \in \mathrm{Th}_{\mathrm{L}}(\Gamma)$

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We write $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\varphi \in \mathrm{Th}_{\mathrm{L}}(\Gamma)$
L is finitary if $\Gamma \vdash_{\mathrm{L}} \varphi$ implies $\Gamma' \vdash_{\mathrm{L}} \varphi$ for finite $\Gamma' \subseteq \Gamma$

Basic notions of AAL II - semantics

Let L be a logic

Fix points of Th_{L} are called theories $\ldots T, S, R$

Th(L) set of theories (*closure system, bounded complete lattice*)

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A Matrix is a pair $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$, where $\boldsymbol{A}$ algebra and $F \subseteq A$

$\mathbb{K}$ class of matrices, then a semantical consequence for $\mathbb{K}$ is

$$\Gamma \models_{\mathbb{K}} \varphi \text{ iff for every } \mathbf{A} = \langle \boldsymbol{A}, F \rangle \in \mathbb{K}, \text{ and } h \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}, \boldsymbol{A}) \\ \text{if } h[\Gamma] \subseteq F \text{ then } h(\varphi) \in F$$

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Given a matrix $\langle \boldsymbol{A}, F \rangle$, then F is an L-Filter if $\vdash_{\mathrm{L}} \subseteq \models_{\{\langle \boldsymbol{A}, F \rangle\}}$

$\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ closure system of L-filters on $\boldsymbol{A}$

$\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}} : P(A) \to P(A)$ is its associated closure operator

Basic notions of AAL III - completeness

Let L be a logic

$\mathbf{MOD}(\mathrm{L}) = \{\langle \boldsymbol{A}, F\rangle \,:\, F \text{ is an L-filter}\}$

$\mathbf{MOD}^*(\mathrm{L}) = \{\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L}) \,:\, \mathbf{A} \text{ is reduced } (\Omega_{\boldsymbol{A}}(F) = \mathrm{Id}_{\boldsymbol{A}})\}$

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Known completeness results

For every logic L we have:

$$\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}(\mathrm{L})} = \models_{\mathbf{MOD}^*(\mathrm{L})}$$

and if L if finitary then L is R(F)SI-complete, i.e.:

$$\vdash_{\mathrm{L}} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}$$

Proof of RSI-completeness

theory T is ∩-prime: $T \neq T_1 \cap T_2$ whenever $T \subsetneq T_1$ and $T \subsetneq T_2$
is completely ∩-prime: $T \neq \bigcap_{i \in I} T_i$ whenever $T \subsetneq T_i$ for every $i \in I$

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Let L be a finitary logic, then it is RSI-complete

1. Take $\Gamma \nvdash_{\mathrm{L}} \varphi$
2. Prove completely $\cap$-prime extension property (CIPEP)
   - Γ can be extended to a completely $\cap$-prime theory T such that $T \nvdash_{\mathrm{L}} \varphi$
3. By the Lindenbaum process applied on T obtain a RSI matrix $\mathbf{A}$ such that $\Gamma \nvDash_{\mathbf{A}} \varphi$

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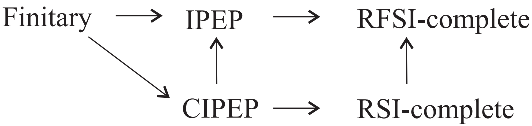

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Semilinear logics

Let L be a logic with implication $\to$

write $\mathbf{MOD}^*(\mathrm{L})^{\to}_{\mathrm{lin}}$ for its *linear models* (order based on $\to$)

L is *semilinear* if

$$\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})^{\to}_{\mathrm{lin}}} \varphi \text{ iff } \varphi \in \mathrm{Th}_{\mathrm{L}}(\Gamma)$$

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If L has implication $\to$ then to prove semilinearity:

1. prove

$$\frac{\Gamma, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi} \text{(SLP)}$$

2. take $\Gamma \nvdash_{\mathrm{L}} \varphi$
3. prove *Linear extension property* (LEP)
   - Γ can be extended to a *linear (either $\varphi \to \psi \in T$ or $\psi \to \varphi \in T$)* theory T such that $T \nvdash_{\mathrm{L}} \varphi$
4. By the Lindenbaum process applied on T we obtain a linear model $\boldsymbol{A}$ such that $\Gamma \not\models_{\boldsymbol{A}} \varphi$

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Proposition 1

If L *has implication* $\to$ *with* (SLP) *then*

1. *T is linear iff T is* $\cap$*-prime*
2. L *has* LEP *iff* L *has* IPEP
3. $\mathbf{MOD}^*(\mathrm{L})^{\to}_{\mathrm{lin}} = \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$

Case of disjunction

Similarly in logics with disjunction connective $\vee$,

and *prime models* $\mathbf{MOD}^*(\mathrm{L})^{\vee}_{\mathrm{prime}}$,

Completeness result:

$$\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})^{\vee}_{\mathrm{prime}}} \varphi \text{ iff } \varphi \in \mathrm{Th}_{\mathrm{L}}(\Gamma)$$

If L **has connective** $\vee$ **then to prove the completeness result:**

1. prove
$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi} \text{ (PCP)}$$
2. take $\Gamma \nvdash_{\mathrm{L}} \varphi$
3. prove *Prime extension property* (PEP)
 - Γ can be extended to a *prime ($\varphi \vee \psi \in T$ iff $\psi \in T$ or $\psi \in T$ for all φ, ψ)* theory T such that $T \nvdash_{\mathrm{L}} \varphi$
4. By the Lindenbaum process applied on T we obtain a prime model $\boldsymbol{A}$ such that $\Gamma \not\models_{\boldsymbol{A}} \varphi$

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Proposition 2

If L *has disjunction* $\vee$ *with* (PCP) *then*

1. *T is prime iff T is* $\cap$*-prime*
2. L *has* PEP *iff* L *has* IPEP
3. $\mathbf{MOD}^*(\mathrm{L})^{\vee}_{\mathrm{prime}} = \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$

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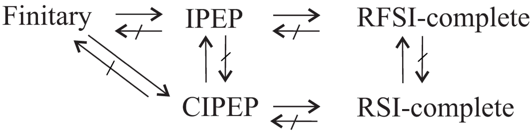
New hierarchy of logics



Surjective evaluations

Given a class of matrices $\mathbb{K}$, then surjective semantical consequence for $\mathbb{K}$ is

$\Gamma \models^{s}_{\mathbb{K}} \varphi$ iff for every $\mathbf{A} = \langle \boldsymbol{A}, F \rangle \in \mathbb{K}$, and surjective $h \in \mathrm{Hom}(\boldsymbol{Fm}_{\mathcal{L}}, \boldsymbol{A})$
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Let $\mathbf{S}$ be the submatrix operator

Observation

For any class of matrices $\mathbb{K}$: $\models_{\mathbb{K}} = \models^{s}_{\mathbf{S}\mathbb{K}}$

Relation to IPEP, CIPEP

Proposition 3 (Characterization of IPEP and CIPEP)

Let L *be protoalgebraic, then*

- L *has* IPEP *iff* $\mathrm{L} = \models^{s}_{\mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}}}$
- L *has* CIPEP *iff* $\mathrm{L} = \models^{s}_{\mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RSI}}}$

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Idea of proof: if $\Gamma \nvdash_{\mathrm{L}} \varphi$ then we get strict surjective

$$h\colon \langle \boldsymbol{Fm}_{\mathcal{L}}, h^{-1}[F]\rangle \to \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}},$$

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Corollary 4

Let L *be protoalgebraic and* $\mathrm{L} = \models_{\mathbb{K}} (= \models^{s}_{\mathbf{S}\mathbb{K}})$*, then*

- *if* $\mathbf{S}\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$*, then* L *has* IPEP
- *if* $\mathbf{S}\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$*, then* L *has* CIPEP

Non-finitary logics with CIPEP

Infinitary Łukasiewicz logic $Ł_\infty$ and Product Π_∞ have CIPEP

Since $Ł_\infty = \models_{\{\langle[0,1]_Ł,\{1\}\rangle\}}$

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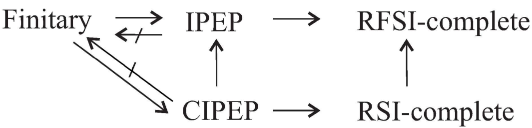

RSI-complete without IPEP

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L is weakly implicative (protoalgebraic) with connective $\to \in \mathcal{L}$

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1. $\langle \boldsymbol{A}, F\rangle$ is reduced (easy because $\rightarrow$), i.e. $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$

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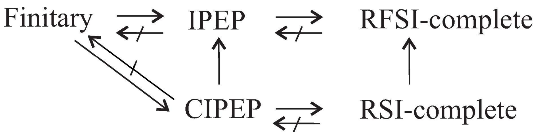
2. F is $\cap$-irreducible in $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, i.e. $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$



Remark

In general we have for any L

$$\text{L has LEP iff } \mathrm{L} = \models_{\mathbf{MOD}^*(\mathrm{L})^{\rightarrow}_{\mathrm{lin}}}$$

$$\text{L has PEP iff } \mathrm{L} = \models_{\mathbf{MOD}^*(\mathrm{L})^{\vee}_{\mathrm{prime}}}$$

yet we have only the direction from left to right

$$\text{L has IPEP then } \mathrm{L} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}}$$

$$\text{L has CIPEP then } \mathrm{L} = \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}$$

Non-RSI completele with IPEP

Let $\mathcal{L} = \{\to\} \cup \{\bar{q} \,:\, q \in (0,1] \cap \mathbb{Q}\}$

For $r \in (0,1] \cap \mathbb{Q}$ define algebra $\boldsymbol{A}_r$ as:

(1) domain is $[0, r]$

(2) $a \to^{\boldsymbol{A}_r} b = q$ if $a \leq b$, and $a \to^{\boldsymbol{A}_r} b = b$ otherwise

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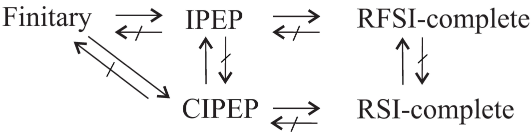
define $\mathbb{K} = \{\langle \boldsymbol{A}_r, \{r\}\rangle : r \in (0,1] \cap \mathbb{Q}\}$ and $\mathrm{L} = \models_{\mathbb{K}}$

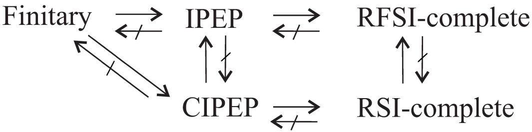
properties

- L is Rasiowa-implicative, semilinear (thus have IPEP)
- $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}} = \{\mathbf{1}\}$, i.e. $\mathrm{L} \neq \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}}$



Conclusion and future work

- Is there proof theoretic characterization for IPEP, CIPEP?
- Are there counter examples with finite signatures?
- Is IPEP equivalent to RFSI-completeness for algebraizable logics?



Finale

Thank you