Riesz MV-algebras and Divisible MV-algebras: logic, analysis and polyhedral geometry

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Riesz MV-algebras (Di Nola, Leuştean, 2014)

- they form a variety,
- categorical equivalence with Riesz Spaces (vector lattices) with strong unit,
- $\mathbb{RMV} = HSP([0,1]_{RMV})$.

Logics

Logic	Algebra	Completeness
$\mathcal{L}$	$Lind_{\mathcal{L}}$ is an MV-algebra	$[0,1]_{MV}$
$\mathbb{Q}\mathcal{L}$	$Lind_{\mathbb{Q}\mathcal{L}}$ is a DMV-algebra	$[0,1] \cap \mathbb{Q}$
$\mathbb{R}\mathcal{L}$	$Lind_{\mathbb{R}\mathcal{L}}$ is a Riesz MV-algebra	$[0,1]_{RMV}$

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Conservative extensions

$\mathbb{Q}\mathcal{L}$ and $\mathbb{R}\mathcal{L}$ are conservative extensions of $\mathcal{L}$

Functional representations

Functional representations

$f : [0,1]^n \to [0,1]$ is a $\mathrm{PWL}_u(\mathbb{Z})$ function if it is continuous and there is a finite set of affine functions $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ with integer coefficients such that for any $(a_1, \ldots, a_n) \in [0,1]^n$ there exists $i \in \{1, \ldots, k\}$ with $f(a_1, \ldots, a_n) = p_i(a_1, \ldots, a_n)$.

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Free Riesz MV-algebra $RMV_n \simeq Lind_{\mathbb{R}\mathcal{L},n}$ [Di Nola, Leuştean 2014]

$RMV_n = \{f_\varphi : [0,1]^n \to [0,1] \mid \varphi$ formula of $\mathbb{R}\mathcal{L}\}=\mathrm{PWL}_u(\mathbb{R})$

Logic and analysis

The logic $\mathbb{R}\mathcal{L}$

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For $r \in [0,1]$ we set $\mathbf{r} := \Delta_r(\theta)$ where $\vdash \theta$

Pavelka-style completeness

If φ is a formula of $\mathcal{R}$Ł, we define:

- the truth degree of φ, by
  $\| \varphi \| = \min\{e(\varphi) \mid e \text{ is an evaluation}\}$,
- the provability degree of φ, by
  $| \varphi | = \max\{r \in [0,1] \mid \vdash \mathbf{r} \rightarrow \varphi\}$,

then

$$| \varphi | = \| \varphi \|$$

Convergence in $\mathbb{R}\mathcal{L}$

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Uniform limit (inspired by [X. Caicedo, LATD'08])

formula φ is the uniform limit of the sequence $(\varphi_n)_n$ in $\mathbb{R}$ if for any $r < 1$ there is k such that for any $n \geq k$: $\vdash \mathbf{r} \rightarrow (\varphi \leftrightarrow \varphi_n)$. We write $\lim_n \varphi_n = \varphi$.

Convergence in $\mathbb{RL}$

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TFAE:

- $\lim_n \varphi_n = \varphi$,
- $\lim_n f_{\varphi_n} = f_\varphi$ (uniform convergence),
- $[\varphi_n] \to [\varphi]$ in $Lind_{\mathbb{RL}}$ (order convergence)

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From $\mathbb{Q}\mathcal{L}$ to $\mathbb{R}\mathcal{L}$

For any formula φ of $\mathbb{R}\mathcal{L}$ there exists a sequence of formulas $(\varphi_n)_n$ of $\mathbb{Q}\mathcal{L}$ such that $\lim_n \varphi_n = \varphi$.

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Completions

The norm-completion of the normed space $(Lind_{\mathbb{RL},n}, \|\cdot\|_u)$ is isometrically isomorphic with $(C([0,1]^n), \|\cdot\|_\infty)$.

Corollary: approximation of continuous functions

For any continuous function $f : [0,1]^n \to [0,1]$ there exists a sequence of formulas $(\varphi_n)_n$ of $\mathbb{RL}$ such that $\lim_n f_{\varphi_n} = f$.

Logic, algebras and tensor product

Categorical adjunctions

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Adjoint functors $(U, \mathcal{D}_{\mathbb{Q}})$, $(U, \mathcal{T}_{\mathbb{R}})$

$\mathbf{MV}_{ss} \xrightarrow{\mathcal{D}_{\mathbb{Q}}} \mathbf{DMV}_{ss} \quad \mathcal{D}_{\mathbb{Q}}(A) = [0,1]_{\mathbb{Q}} \otimes A$

$\mathbf{MV}_{ss} \xrightarrow{\mathcal{T}_{\mathbb{R}}} \mathbf{RMV}_{ss} \quad \mathcal{T}_{\mathbb{R}}(A) = [0,1] \otimes A$ [S.L., I. Leuştean, 2016]

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Representation of free algebras

- $Lind_{\mathbb{Q}\mathcal{L}} \simeq [0,1]_{\mathbb{Q}} \otimes Lind_{\mathcal{L}}$, $\quad DMV_n \simeq [0,1]_{\mathbb{Q}} \otimes MV_n$

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- $Lind_{\mathbb{R}\mathcal{L}} \simeq [0,1] \otimes Lind_{\mathcal{L}}$, $\quad RMV_n \simeq [0,1] \otimes MV_n$

Finitely presented structures

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Tensor functors between finitely presented structures

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$\mathbf{MV_{fp}}$, $\mathbf{DMV_{fp}}$ and $\mathbf{RMV_{fp}}$ are the full subcategories of finitely presented MV, DMV and Riesz MV-algebras.

We call MV-elements the elements of $\iota_A(A)$ (in either $[0,1] \otimes A$ or $[0,1]_Q \otimes A$).

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Category	Objects	Morphisms
$\mathbf{DMV_{fp}}'$	$D = [0,1]_{\mathbb{Q}} \otimes A$, $A \in \mathbf{MV_{fp}}$	DMV-maps that preserve the MV-elements
$\mathbf{RMV_{fp}}'$	$R = [0,1] \otimes A$, $A \in \mathbf{MV_{fp}}$	RMV-maps that preserve the MV-elements

$\mathbf{MV_{fp}}$, $\mathbf{DMV_{fp}}'$ and $\mathbf{RMV_{fp}}'$ are equivalent.

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Remark: Any finitely presented DMV-algebra is (isomorphic to one) of the form $[0,1]_Q \otimes A$, i.e. $\mathcal{D}_\otimes$ is essentially surjective.
This is not the case for finitely presented Riesz MV-algebras! Pres

Logic and polyhedra

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On DMV-algebras:

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Categories of polyhedra

For $P \subseteq [0,1]^n$, $Q \subseteq [0,1]^m$,

Category	Object	Morphism
$Pol^{\mathbb{R}}_{[0,1]}$	P polyhedron	$\lambda : P \to Q$, $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda_i \in RMV_n$
$RatPol^{\mathbb{Q}}_{[0,1]}$	P rational polyhedron	$\lambda : P \to Q$, $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda_i \in DMV_n$
$RatPol^{\mathbb{Z}}_{[0,1]}$	P rational polyhedron	$\lambda : P \to Q$, $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda_i \in MV_n$

Finitely presented structures and polyhedra

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The following categories are dual:

- $\mathbf{MV_{fp}}$ and $RatPol^{\mathbb{Z}}_{[0,1]}$ [Marra, Spada 2013]
- $\mathbf{DMV_{fp}}$ and $RatPol^{\mathbb{Q}}_{[0,1]}$
- $\mathbf{RMV_{fp}}$ and $Pol^{\mathbb{R}}_{[0,1]}$ [Di Nola, Lenzi, Vitale, 2016]

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- $\mathbf{RMV_{fp}}$ and $Pol^{\mathbb{R}}_{[0,1]}$ [Di Nola, Lenzi, Vitale, 2016]

Moreover,

$DMV_n \mid_P$ is finitely presented and projective iff P is a retract of $[0,1]^n$,

The duality can be extended to polyhedra in the whole space.

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For $P \subseteq \mathbb{R}^n$, $Q \subseteq \mathbb{R}^m$,

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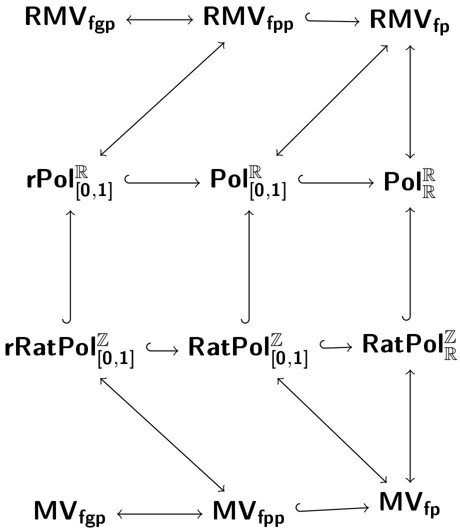

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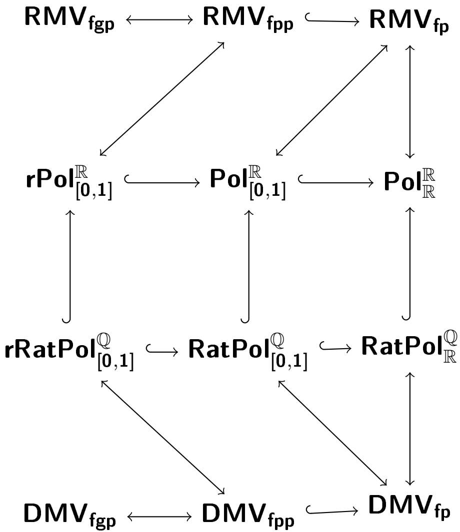

Tensor product and polyhedra

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Hom-functors

If $\mathbf{C_n} = [0,1]^n$ and $\mathbf{H}_{\mathbb{Z}}(-, \mathbf{C_n}) : \mathbf{RatPol}^{\mathbb{Z}}_{[0,1]} \to Set$ is the Hom-functor and $\mathbf{H}_{\mathbb{Z}}(P, \mathbf{C}_n) = \prod_{i=1}^{n} MV_m \mid_P$ is a semisimple MV-algebra.

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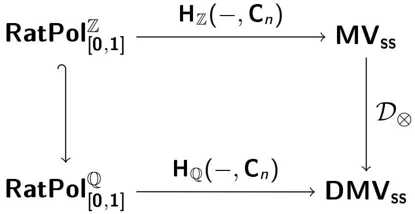
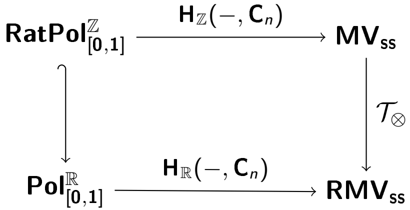
We define

$$\mathbf{H}_{\mathbb{Z}}(-, \mathbf{C}_n) : \mathbf{RatPol}^{\mathbb{Z}}_{[0,1]} \to \mathbf{MV_{ss}},$$
$$\mathbf{H}_{\mathbb{Q}}(-, \mathbf{C}_n) : \mathbf{RatPol}^{\mathbb{Q}}_{[0,1]} \to \mathbf{DMV_{ss}},$$
$$\mathbf{H}_{\mathbb{R}}(-, \mathbf{C}_n) : \mathbf{Pol}^{\mathbb{R}}_{[0,1]} \to \mathbf{RMV_{ss}}$$

The following diagrams are commutative:



- $\mathbf{H}_{\mathbb{Q}}(P, \mathbf{C}_n) \simeq \mathcal{D}_{\otimes}(\mathbf{H}_{\mathbb{Z}}(P, \mathbf{C}_n))$



- $\mathbf{H}_{\mathbb{R}}(P, \mathbf{C}_n) \simeq \mathcal{T}_{\otimes}(\mathbf{H}_{\mathbb{Z}}(P, \mathbf{C}_n))$

Categories of presentations

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Category	Objects	Morphisms
ThMV	(X, I) X set, $I \subseteq MV_X$ principal ideal	$\psi : (X, I) \to (Y, J)$ $\psi : MV_X \to MV_Y$ $\psi(I) \subseteq J$
ThDMV$'$	(X, I) X set, $I \subseteq DMV_X$ principal ideal	$\psi : (X, I) \to (Y, J)$ $\psi : DMV_X \to DMV_Y$ $\psi(I) \subseteq J$, $\psi(X) \subseteq MV_Y$
ThRMV$^{mv'}$	(X, I) X set, $I \subseteq DMV_X$ principal ideal $I = id(f)$, $f \in DMV_X$	$\psi : (X, I) \to (Y, J)$ $\psi : DMV_X \to DMV_Y$ $\psi(I) \subseteq J$, $\psi(X) \subseteq MV_Y$

Pres

$\mathcal{DT}$: **ThMV**$\to$ **ThDMV**$'$

- Objects: $\mathcal{DT}(X, I) = (X, id(I)_{DMV})$;
- Arrows: for $\sigma : (X, I) \to (Y, J)$, $\mathcal{DT}(\sigma)$ is defined by $\sigma_D(x)) = \sigma(x)$ for any $x \in X$.

$\mathcal{RT}$:**ThMV**$\to$ **ThRMV**$^{\mathbf{mv}'}$

defined the same

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ThMV **ThDMV**$'$ and **ThRMV**$^{\mathbf{mv}'}$ are categorical equivalent.

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then there exists a formula β of $\mathcal{L}$ s.t. $\alpha \in \mathit{Thm}(\beta, \mathcal{L})$

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this is no longer true for $\mathbb{R}\mathcal{L}$.

Thank you!