Strong standard completeness of BL

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LATD 2016

I would like to thank my supervisor Professor Ian Hodkinson for his ideas and scrutinising readings.

Outline

Introduction

1. We want to prove strong standard completeness for an axiomatic system of logic BL with respect to continuous t-norms as well as axiomatic systems of Łukasiewicz logic and Product logic with respect to Łukasiewicz and Product t-norms, resp.
2. We need to add an infinitary rule to Hájek proof system as examples show that no finitary proof system can be strongly standard complete.
3. We also add a new Prelin rule to replace Hajek's prelinearity axiom to get around the lack of a deduction theorem.

Introduction

Other approaches involved extending the language either by

1. introducing a connective (Montagna)
2. introducing more truth constants (Cintula and most recently the work of Vidal, Bou, Esteva and Godo)

In all cases infinitary rules were employed to show strong standard completeness for the given logics.

We do not need to extend the language.

Fix a countably infinite set *Var* of propositional atoms. Let *Fm* be the set of all formulas written with atoms from *Var* and the nullary connective $\bar{0}$ using binary connectives $\&, \rightarrow$. The semantics for atoms in *Var* is defined by a function $V : Var \rightarrow [0, 1]$. Before we define the semantics for all formulas, we will introduce a continuous t-norm.

Preliminaries

A continuous t-norm $\star : [0,1]^2 \to [0,1]$ is a function which is associative, commutative, monotonous and satisfying boundary condition: $x \star 1 = x$ for every $x \in [0,1]$. Its residuum $\Rightarrow : [0,1]^2 \to [0,1]$ satisfies for all $x, y, z \in [0,1]$:

$$x \star z \leq y \leftrightarrow z \leq x \Rightarrow y$$

1. Łukasiewicz t-norm and its residuum: $x \star y = \max\{0, x + y - 1\}$ and $x \Rightarrow y = 1 - x + y$ if $y < x$ and $x \Rightarrow y = 1$ otherwise.
2. Product t-norm and its residuum: $x \star y = xy$ and $x \Rightarrow y = y/x$ if $y < x$ and $x \Rightarrow y = 1$ otherwise.

Preliminaries

The semantics for all formulas from Fm is defined inductively from $V : Var \to [0,1]$:

1. $V_\star(\bar{0}) = 0$,
2. $V_\star(\alpha \& \beta) = V_\star(\alpha) \star V_\star(\beta)$ for $\alpha, \beta \in Fm$,
3. $V_\star(\alpha \to \beta) = V_\star(\alpha) \Rightarrow V_\star(\beta)$ for $\alpha, \beta \in Fm$.

The set Fm with syntax and semantics defined as above is BL logic. If $\star$ is Łukasiewicz or Product, then we deal with Łukasiewicz or Product logics, resp.

Outline

Proof systems

BL-Axioms (A1)-(A7), axioms ($\neg\neg$), (Π1), (Π2)

We will use the following axioms and inference rules.

(A1) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

(A2) $\alpha \& \beta \to \alpha$

(A3) $\alpha \& \beta \to \beta \& \alpha$

(A4) $\alpha \& (\alpha \to \beta) \to \beta \& (\beta \to \alpha)$

(A5a) $(\alpha \to (\beta \to \gamma)) \to (\alpha \& \beta \to \gamma)$

(A5b) $(\alpha \& \beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$

(A7) $\bar{0} \to \alpha$

($\neg\neg$) $((\alpha \to \bar{0}) \to \bar{0}) \to \alpha$

(Π1) $\neg\neg\gamma \to ((\alpha \& \gamma \to \beta \& \gamma) \to (\alpha \to \beta))$

(Π2) $\alpha \wedge \neg\alpha \to \bar{0}$

Proof systems

Inference Rules

$$\frac{\alpha, \alpha \to \beta}{\beta} \tag{MP}$$

$$\frac{T \cup \{\alpha \to \beta\} \vdash \varphi, T \cup \{\beta \to \alpha\} \vdash \varphi}{T \vdash \varphi} \tag{Prelin}$$

Logic BL

Our axiomatic system for logic BL is BL-axioms with rules (MP) and (Prelin).

Łukasiewicz Logic

Our axiomatic system for Łukasiewicz logic is BL-axioms with axiom ($\neg\neg$) and rules (MP) and (Prelin).

Product Logic

Our axiomatic system for Product logic is BL-axioms with axioms ($\Pi 1$), ($\Pi 2$) and rules (MP) and (Prelin).

Definition of Proof

Let $T \in \wp(Fm)$. A *proof from* T is a sequence of pairs $(T_i, \varphi_i) \in \wp(Fm) \times Fm (i \leq \xi)$ for some ordinal ξ, where for each $i \leq \xi$, $T_i \subseteq Fm$ is finite, $\varphi_i \in Fm$, and letting $\mathcal{H} = \{(T_j, \varphi_j) : j < i\}$, at least one of the following holds:

1. φ_i is a BL-axiom,
2. $\varphi_i \in T \cup T_i$,
3. there is a formula χ such that $(T_i, \chi), (T_i, \chi \to \varphi_i) \in \mathcal{H}$,
4. there are formulas α, β such that $(T_i \cup \{\alpha \to \beta\}, \varphi_i), (T_i \cup \{\beta \to \alpha\}, \varphi_i) \in \mathcal{H}$.

We call (T_i, φ_i) a *proof element*. The set T_i comprises assumptions sufficient to prove φ_i. For a formula φ, a *proof of* φ *from* T is a proof from T ending in $(\emptyset, \varphi)$. If there is a proof of φ from T, we write $T \vdash_{BL} \varphi$ and if T is $\emptyset$, $\vdash_{BL} \varphi$.

Outline

General Soundness

Definition

Let $\models_{BL} \subseteq \wp(Fm) \times Fm$ be given by $T \models_{BL} \varphi$ iff for every continuous t-norm $\star$ and all evaluations $V : PROP \to [0,1]$, if $V_\star(\theta) = 1$ for every $\theta \in T$, then $V_\star(\varphi) = 1$. If $T = \emptyset$, we denote $T \models_{BL} \varphi$ as $\models_{BL} \varphi$.

Definition

We say that the axiomatic system of BL logic is *generally sound* if $\vdash_{BL} \subseteq \models_{BL}$.

Theorem

The axiomatic system of BL logic is generally sound.

Outline

Strong Standard Completeness

Definition

1. We say that the axiomatic system of BL logic is *weakly standard complete* if the following holds. For every $\psi \in Fm$, if $\models_{BL} \psi$, then $\vdash_{BL} \psi$.
2. We say that the axiomatic system of BL logic is *finitely strongly standard complete* if the following holds. For every $\psi \in Fm$ and every finite $T \in \wp(Fm)$, if $T \models_{BL} \psi$, then $T \vdash_{BL} \psi$.
3. We say that the axiomatic system of BL logic is *strongly standard complete* if $\models_{BL} \subseteq \vdash_{BL}$.

Outline

Extending axiomatic systems with (Inf)

Example

BL logic. Let $T = \{p \to q^n : n < \omega\}$ with $p, q \in PROP$. Then $T \models_{BL} p \to p\&q$, but $T \not\vdash_{BL} p \to p\&q$.

$$\frac{\varphi \vee (\alpha \to \beta^n) : n < \omega}{\varphi \vee (\alpha \to \alpha\&\beta)} \tag{Inf}$$

Logic BL

Our extended axiomatic system for logic BL is BL-axioms with rules (MP), (Prelin) and (Inf).

Definition of Proof

Let $T \in \wp(Fm)$. A *proof from* T is a sequence of pairs $(T_i, \varphi_i) \in \wp(Fm) \times Fm (i \leq \xi)$ for some ordinal ξ, where for each $i \leq \xi$, $T_i \subseteq Fm$ is finite, $\varphi_i \in Fm$, and letting $\mathcal{H} = \{(T_j, \varphi_j) : j < i\}$, at least one of the following holds:

1. φ_i is a BL-axiom,
2. $\varphi_i \in T \cup T_i$,
3. there is a formula χ such that $(T_i, \chi), (T_i, \chi \to \varphi_i) \in \mathcal{H}$,
4. there are formulas α, β such that $(T_i \cup \{\alpha \to \beta\}, \varphi_i), (T_i \cup \{\beta \to \alpha\}, \varphi_i) \in \mathcal{H}$,
5. there are formulas φ, α, β such that $\varphi_i = \varphi \vee (\alpha \to \alpha \& \beta)$ and $(T_i, \varphi \vee (\alpha \to \beta^n)) \in \mathcal{H}$ for each $n < \omega$.

If there is a proof of φ from T, we write $T \vdash \varphi$ and if T is $\emptyset$, $\vdash \varphi$.

Outline

Theorem

The axiomatic system of BL logic is strong standard complete with respect to continuous t-norms.

Sketch of a proof.

Let $T_0 \in \wp(Fm)$. Assume $T_0 \nvdash \psi$. We show that $T_0 \not\models \psi$.

1. We extend T_0 to T^* with some properties.
2. We define special relations between formulas: pre-clusters and blocks.
3. We define clusters as subsets of a set of blocks: if there is a pre-cluster relation between blocks, then they are in the same cluster. There are two types of clusters: Łukasiewicz and Product.

A prelinearly and deductively closed theory

Firstly, we extend T_0 to T^* with some properties.

Definition

We say that $T \in \wp(Fm)$ is

1. *prelinearly closed* if for each $\varphi, \psi \in Fm$, we have $\varphi \to \psi \in T^*$ or $\psi \to \varphi \in T^*$,
2. *deductively closed* if $\varphi \in T$ whenever $T \vdash \varphi$.

Theorem

Let $T_0 \in \wp(Fm)$ and $\psi_0 \in Fm$, and assume $T_0 \nvdash \psi_0$. Then there is a prelinearly and deductively closed theory $T^* \supseteq T_0$ with $T^* \nvdash \psi_0$.

Pre-clusters

Secondly, we define special relations between formulas.

Definition

For $\alpha, \beta \in Fm$, we define:

1. $\alpha \leq \beta$ iff $T^* \vdash \alpha \to \beta$,
2. $\alpha < \beta$ iff $\alpha \leq \beta$ and $\beta \nleq \alpha$,
3. $\alpha \sim \beta$ iff there is $n < \omega$ such that $\beta^n \leq \alpha \leq \beta$ or $\alpha^n \leq \beta \leq \alpha$ (pre-clusters),
4. $\alpha \approx \beta$ iff $\alpha \leq \beta$ and $\beta \leq \alpha$ (blocks).

Properties of pre-clusters

Lemma

Let $\alpha, \beta \in \mathcal{F}$. *If* $\alpha \sim \beta$, *then* $\alpha \& \beta \sim \alpha$.

Lemma

Let $\alpha, \beta \in \mathcal{F}$. *If* $\alpha < \beta$ *and* $\alpha \not\sim \beta$, *then* $\alpha \& \beta \approx \alpha \approx \beta \to \alpha$.

Lemma

Let $\alpha, \beta \in \mathcal{F}$. *If* $\alpha \sim \beta$ *and* $\alpha < \beta$, *then* $\beta \to \alpha \sim \alpha$.

Clusters

We denote the $\approx$-equivalence class $\alpha/_{\approx}$ as $\hat{\alpha}$ and we define $\mathfrak{A} = Fm/_{\approx}$, $\hat{0} = \bar{0}/_{\approx}, \hat{1} = \bar{1}/_{\approx}$. We define a binary relation $\leq$ on $\mathfrak{A}$: $\hat{\alpha} \leq \hat{\beta}$ iff $\alpha \leq \beta$. We define binary functions $\star$ and $\Rightarrow$ on $\mathfrak{A}$: $\hat{\alpha} \star \hat{\beta} = \widehat{\alpha \& \beta}$ and $\hat{\alpha} \Rightarrow \hat{\beta} = \widehat{\alpha \to \beta}$. We also define $\hat{\alpha}^{\star(0)} = \hat{1}$ and inductively $\hat{\alpha}^{\star(n+1)} = \hat{\alpha}^{\star(n)} \star \hat{\alpha}$.

Definition

We define a cluster $C \subseteq \mathfrak{A}$ as follows: $\hat{\alpha}, \hat{\beta} \in C$ iff $\alpha \sim \beta$. We define the order of clusters C, D: $C < D$ iff for all $x \in C$ and all $y \in D$, $x < y$.

Łukasiewicz and Product clusters

We say that a cluster C is Łukasiewicz iff there is $x \in C$ such that $x^{\star(2)} = x$ and we call x an idempotent, otherwise it is Product.

Sketch of the Proof

1. We assumed that $T_0 \nvdash \psi_0$ and constructed T^*.
2. From $T^* \nvdash \psi_0$ it follows that $\hat{\psi_0} < \hat{1}$.
3. We select endpoints in [0,1] for each cluster in the same order as clusters.
4. For each cluster $C < \{\hat{1}\}$, we constructed a linearly ordered Abelian group and showed that it is Archimedean. Thus by Hölder's theorem, we could map it onto reals. Then we mapped reals onto the real intervals with the cluster's endpoints and with Łukasiewicz or Product linearly scaled t-norms.
5. By decomposition theorem, we constructed the ordinal sum $\star_{LP}$ of a family of t-norms and an evaluation $V : PROP \rightarrow [0,1]$ such that $V_{\star_{LP}}(\psi) < 1$.

Łukasiewicz and Product logics

To prove that the axiomatic systems for Łukasiewicz and Product logics are strongly standard complete, we proceed as in the proof above, but we showed that

1. For Łukasiewicz logic, there are two distinct Łukasiewicz clusters C with $\hat{0} \in C$ and $\{\hat{1}\}$.
2. For Product clusters, there are at most three clusters: two Łukasiewicz $\{\hat{0}\}, \{\hat{1}\}$ and at most one Product C such that $\{\hat{0}\} < C < \{\hat{1}\}$.