# Strong Completeness for continuous t-norms

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**Fuzzy Logic BL.** Fuzzy logic is a study of valid reasoning based on degrees of truth ranging over the interval of [0,1]. Axiomatic systems for logics in general enable from a selected formulas (axioms) derive new formulas (theorems) using inference (derivation) rules. Showing soundness of those systems that is that they prove only true theorems is achievable and required for any axiomatic systems and this is called a semantic approach. However, showing standard completeness, a syntactic approach, that is that any true theorem is provable is much harder. Even harder if we extend standard completeness to strong standard completeness, that is standard completeness over all theories (sets of formulas), i.e. given any theory and any formula if from the theory we can show that the given formula is true provided that all formulas in the theory are true, then from this theory we can prove the formula using the axioms and inference rules. If strong standard completeness can proved for an axiomatic system, then we semantic and syntactic approaches coincide and whatever formula is true can proved and vice verse.

Our interest in this paper concentrates around axiomatic systems of one of propositional fuzzy logic called BL introduced by Hájek in [6] as well as Product and Łukasiewicz logics. We will demonstrate that all enjoy strong standard completeness property.

Formulas of BL and the other fuzzy logics are constructed from propositional atoms joined by nullary connective $\bar{0}$ (representing falsity) and binary connectives: strong conjunction & and implication $\rightarrow$. The other connectives can be defined using $\bar{0}, \&, \rightarrow$: for any formulas $\varphi, \theta$, $\varphi \wedge \theta$ as $\varphi \& (\varphi \rightarrow \theta)$, $\varphi \vee \theta$ as $((\varphi \rightarrow \theta) \rightarrow \theta) \wedge ((\theta \rightarrow \varphi) \rightarrow \varphi)$, $\neg\varphi$ as $\varphi \rightarrow \bar{0}$ and $\bar{1}$ (representing truth) as $\bar{0} \rightarrow \bar{0}$. To define the semantics of BL, we need to introduce a t-norm. A t-norm $\star : [0,1]^2 \rightarrow [0,1]$ is a commutative, associative and non-decreasing function satisfying a boundary condition: for any $x \in [0,1]$, $x \star 1 = x$. The residuum of $\star$, $\Rightarrow$: $[0,1]^2 \rightarrow [0,1]$ is defined as follows: for any $x, y \in [0,1]$, $x \Rightarrow y = \sup\{z \in [0,1] : x \star z \leq y\}$. Now, given the evaluation of propositional atoms $V$ into [0,1], $V_\star$ of formulas into [0,1] is (1) $V_\star(p) = V(p)$ for a propositional atom $p$, (2) $V_\star(\bar{0}) = 0$, and for any two formulas $\varphi, \chi$ of BL, (3) $V_\star(\varphi \& \chi) = V_\star(\varphi) \star V_\star(\chi)$, (4) $V_\star(\varphi \rightarrow \chi) = V_\star(\varphi) \Rightarrow V_\star(\chi)$, (5) $V_\star(\varphi \wedge \chi) = \min\{V_\star(\varphi), V_\star(\chi)\}$, and (6) $V_\star(\varphi \vee \chi) = \max\{V_\star(\varphi), V_\star(\chi)\}$. The semantics of the strong conjunction and implication for fuzzy logics (1) Łukasiewicz and (2) Product is given respectively by the following t-norms and their residua: for any $x, y \in [0,1]$, (1) Łukasiewicz t-norm $x \star y = \max\{0, x+y-1\}$ and its residuum $x \Rightarrow y = \min\{1, 1-x+y\}$, (2) Product t-norm $x \star y = x \cdot y$ and its residuum $x \Rightarrow y = y/x$ if $y < x$, and $x \Rightarrow y = 1$, otherwise. The decomposition theorem (see [6], [8]) demonstrates that a continuous t-norm is the ordinal sum of a family of t-norms isomorphic to either Łukasiewicz t-norm or Product t-norm, which we are using in the proof of the strong completeness theorem.

**Axiomatic system of BL and its extensions.** Hájek defined an axiomatic system for fuzzy logic BL in [6]. It consists of seven axioms, listed below, and an inference rule Modus Ponens.

| | | | |
|---|---|---|---|
| (A1) | $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$ | (A5a) | $(\alpha \to (\beta \to \gamma)) \to (\alpha \& \beta \to \gamma)$ |
| (A2) | $\alpha \& \beta \to \alpha$ | (A5b) | $(\alpha \& \beta \to \gamma) \to (\alpha \to (\beta \to \gamma))$ |
| (A3) | $(\alpha \& \beta) \to (\beta \& \alpha)$ | (A6) | $((\alpha \to \beta) \to \gamma) \to (((\beta \to \alpha) \to \gamma) \to \gamma)$ |
| (A4) | $(\alpha \& (\alpha \to \beta)) \to (\beta \& (\beta \to \alpha))$ | (A7) | $\bar{0} \to \alpha$ |

Given formulas $\varphi, \chi, \theta$, the following axioms extend BL to stronger logics of (1) Łukasiewicz by axiom ($\neg\neg$): $((\varphi \to \bar{0}) \to \bar{0}) \to \varphi$, (2) Product by axioms ($\Pi 1$): $\neg\neg\chi \to ((\varphi \& \chi \to \theta \& \chi) \to (\varphi \to \theta))$, and ($\Pi 2$): $\varphi \wedge \neg\varphi \to \bar{0}$. It has been proven that the axiomatic systems for BL, Łukasiewicz and Product logics are weakly standard complete (strong standard complete over empty theories) and it has been shown by counterexamples that they are not strongly complete (see [1], [2], [3], [4], [5], [6], [7]).

In our paper we will extend these axiomatic systems of BL, Łukasiewicz and Product by additional rules to achieve strong standard completeness for continuous t-norms. These rules are the infinitary rule (Inf) and the prelinearity rule (Prelin) as defined below. Let $T$ be a set of formulas, $\varphi, \alpha, \beta$ are formulas and $\vdash$ denotes the extension of $\vdash_{BL}$, the BL proof system,

$$\frac{\varphi \vee (\alpha \to \beta^n) : n < \omega}{\varphi \vee (\alpha \to \alpha \& \beta)} \tag{Inf}$$

$$\frac{T \cup \{\alpha \to \beta\} \vdash \varphi, T \cup \{\beta \to \alpha\} \vdash \varphi}{T \vdash \varphi} \tag{Prelin}$$

Following the decomposition theorem, if we take two numbers $x, y \in [0, 1]$ and there is no interval of [0,1] isomorphic to either Lukasiewicz or Product t-norm to which they belong, then $x \star y = \min\{x, y\}$. Our infinitary rule illustrates this property for formulas of BL. We also drop axiom (A6) from these extended systems as it can be derived from other axioms using (Prelin) rule.

**Strong Completeness.** As aforementioned, it has been shown that the axiomatic systems for BL, Łukasiewicz and Product logics failed the strong standard completeness in [9], [6] page 75 and in [4], respectively. However, when we allow infinitary rules, the picture changes. In paper [9], to achieve a strong standard completeness result for continuous t-norms, Montagna extended the language of BL by introducing a new unary connective $*$ defined as follows: for a formula $\varphi, \varphi^*$ is the greatest idempotent below $\varphi$, and he introduced an infinitary rule to the axiomatic system of his extended BL. The advantage of our proof of a strong completeness for continuous t-norms is that we do not extend the language of BL. Our infinitary rule is thus different since Montagna's uses this additional connective. Also, we show that we can further extend this axiomatic system by axioms $(\neg\neg)$ and $(\Pi1), (\Pi2)$ to achieve strong standard completeness for Łukasiewicz and Product t-norms, respectively.

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# References

[1] C. C. Chang. A New Proof of the Completeness of the Lukasiewicz Axioms. *Transactions of the American Mathematical Society*, 93(1):74–80, October 1959.

[2] R. Cignoli, F. Estava, L. Godo, and A. Torrens. Basic Fuzzy Logic is the logic of continuous t-tnorms and their residua. *Soft Computing*, 4:106–112, 2000.

[3] P. Cintula, P. Hájek, and C. Noguera. *Handbook of mathematical fuzzy logic*. College Publications, London, 2011.

[4] P. Cintula and M. Navara. Compactness of fuzzy logics. *Fuzzy sets and systems*, 143:59–73, 2004.

[5] P. Hájek. Basic fuzzy logic and BL-algebras. *Soft Computing*, 2:124–128, 1998.

[6] P. Hájek. *Methamathematics of Fuzzy Logic*. Kluwer Academic Publishers, Dordrecht, 1998.

[7] P. Hájek, L. Godo, and F. Estava. A complete many-valued logic with product conjunction. *Archive for Mathematical Logic*, 35:191–208, 1996.

[8] G. Metcalfe, N. Olivetti, and D. M. Gabbay. *Proof Theory for Fuzzy Logic*. Springer, 2009.

[9] F. Montagna. Notes on Strong Completeness in Łukasiewicz, Product and BL Logics and in Their First-Order Extensions. *Algebraic and Proof-theoretic Aspects of Non-classical Logics*, pages 247–274, 2007.