Right-divisible residuated posets, semilattices and residuated Heyting algebras

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- Quasigroups and groups
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Groups and quasigroups

Definition

A ***quasigroup*** $(A, \cdot, \backslash, /)$ is a set with 3 binary operations such that for all $x, y, z \in A$

$$xy = z \iff x = z/y \iff y = x\backslash z$$

i.e., one can solve **all equations** with no repeated variables

Quasigroups form a **variety** defined by the identities

$$(x/y)y = x = xy/y \quad \text{and} \quad y(y\backslash x) = x = y\backslash yx$$

A quasigroup with **associative** $\cdot$ is **term-equivalent to a group**:

$$1 = y/y \quad \text{and} \quad x^{-1} = (y/y)/x$$

Hint: $xy/z = x((y/z)z)/z = (x(y/z))z/z = x(y/z)$
hence $x = xy/y = x(y/y)$ and therefore $x\backslash x = y/y$

Residuated binars and semigroups

Definition

A ***residuated binar*** $(A, \leq, \cdot, \backslash, /)$ is a poset $(A, \leq)$ with 3 binary operations such that for all $x, y, z \in A$

$$xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$$

i.e., one can solve **simple inequalities**

Residuated binars are defined (relative to posets) by the **inequations**

$$(x/y)y \leq x \leq xy/y \quad \text{and} \quad y(y\backslash x) \leq x \leq y\backslash yx$$

Farulewski 2005: The universal theory of residuated binars is **decidable**

Definition

A ***residuated semigroup*** is an **associative** residuated binar

If the poset is an **antichain**, then any **residuated semigroup is a group**!
$\Rightarrow$ residuated semigroups are **generalizations of groups** (replace $=$ by $\leq$)

Residuated lattices and GBL-algebras

Definition

A **residuated lattice** $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated ℓ-monoid

i.e., a lattice $(A, \wedge, \vee)$ and a residuated semigroup with unit

They are the algebraic semantics of **substructural logic**

The **equational theory** of residuated lattices is **decidable**

Definition

A residuated lattice is **divisible** if $x \leq y \implies x = y(y\backslash x) = (x/y)y$

Also called a **generalized Basic Logic** algebra (GBL-algebra)

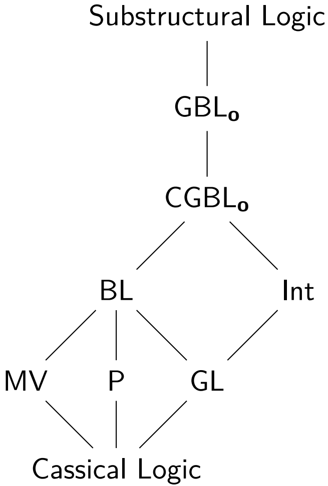
Open problem: Is the equational theory of GBL-algebras **decidable**?

Bounded GBL-algebras

- Expand with a bottom element **0** to get $\text{GBL}_{\mathbf{o}}$-algebras
- They are a variety that includes all **Heyting algebras** and **MV-algebras**
- Have **distributive** lattice reducts (like HA and MV) **[J. - Tsinakis 2002]**
- All **finite** GBL-algebras are **commutative** and **integral** **[J. - Montagna 2006]**
- All **finite** GBL-algebras are **poset products** of MV-chains **[J. - Montagna 2009]**

- **Open problem**: develop a **structure theory** for $\text{GBL}_{\mathbf{o}}$-algebras

Some Substructural Logics



Right-residuated binars

GBL-algebras fairly complicated, so consider **simpler** algebras

Definition

A ***right-residuated binar*** $(A, \leq, \cdot, /)$ is a poset $(A, \leq)$ with 2 binary operations such that for all $x, y, z \in A$

$$xy \leq z \iff x \leq z/y$$

Therefore $\cdot, /$ are **order-preserving in the left argument**:

Let $x \leq y$. Then $yz \leq yz \iff y \leq yz/z \implies x \leq yz/z \iff xz \leq yz$

Similarly $x/z \leq x/z \iff (x/z)z \leq x \implies (x/z)z \leq y \iff x/z \leq y/z$

It would be nice if $\leq$ is **definable** from the algebraic operations

Right-divisible residuated binars

Theorem

The following are equivalent in any right-residuated binar.

(i) *For all* x, y $(x \le y \iff \exists u(x = uy))$

(ii) *For all* x, y $(x \le y \iff x = (x/y)y)$ *(i.e. right divisibility).*

(iii) *The identities* $(y/y)x = x$ *and* $(y/x)x = (x/y)y$ *hold.*

Proof: (i)⇒(ii): Suppose $x \le y \iff \exists u(x = uy)$ holds.

Assuming $x \le y$ one obtains $uy = x \le x$ for some u, hence $u \le x/y$.

Since $\cdot$ is order preserving in the left argument, we have $x = uy \le (x/y)y$.

The reverse inequality $(x/y)y \le x$ holds in any right-residuated binar, so we conclude that $x \le y$ implies $x = (x/y)y$.

Conversely, if $x = (x/y)y$ holds, then $\exists u(x = uy)$, whence the first condition implies $x \le y$.

(ii) Divisibility iff (iii) $(y/y)x = x$ and $(y/x)x = (x/y)y$

(ii)$\Rightarrow$(i) since we can take $u = x/y$.

(ii)$\Rightarrow$(iii): Assume that $x \le y \iff x = (x/y)y$ for all x, y.

Since $x \le x$, we get $x = (x/x)x$.

We always have $x \le xy/y$, hence $xy \le (xy/y)y$ holds.

The reverse inequality is also true in general, so $xy = (xy/y)y$.

From the assumption it follows that $xy \le y$.

Therefore we have $x \le y/y$ as an identity, hence $x/x \le y/y$.

Interchanging x, y proves $x/x = y/y$.

Multiplying by x on the right we get $x = (x/x)x = (y/y)x$.

(ii) Divisibility iff (iii)$(y/y)x = x$ and $(y/x)x = (x/y)y$

To prove $(y/x)x = (x/y)y$: Recall $(x/y)y \le x$, and use the assumption with x replaced by $(x/y)y$ and y replaced by x to get $(x/y)y = ((x/y)y/x)x$.

As in the proof of the first identity, we have $xy \le y$.

Dividing and multiplying by z on both sides gives the identity $(xy/z)z \le (y/z)z$.

Now replace x by x/y and z by x to see that $((x/y)y/x)x \le (y/x)x$.

It follows that $(x/y)y \le (y/x)x$, and interchanging x, y proves the identity.

(iii)$\Rightarrow$(ii): Assume the identities $(y/y)x = x$ and $(y/x)x = (x/y)y$ hold.

Want to prove: for all x, y ($x \le y \iff x = (x/y)y$) (i.e. divisibility)

(iii)$(y/y)x = x$ and $(y/x)x = (x/y)y \;\Rightarrow\; (ii)$*Divisibility*

Let $x \leq y$. Then $(y/y)x \leq y$, hence $y/y \leq y/x$.

Multiply by x on the right to get $x = (y/y)x \leq (y/x)x = (x/y)y$.The reverse inequality follows from right-residuation, whence $x = (x/y)y$.

Again, assume the two identities of (iii) holds, and let $x = (x/y)y$.

By right-residuation we have $(y/x)x \leq y$, so we

deduce $(x/y)y \leq y$ from the second identity.

Since we started with $x = (x/y)y$, we conclude that $x \leq y$.

Right-divisible unital residuated binar

The identities for divisibility are $(y/y)x = x$ and $(y/x)x = (x/y)y$

So y/y is a left unit, and the proof of (ii)$\Rightarrow$(iii) showed $x \le y/y$

Hence y/y is the top element of the poset, denoted by 1

A **right-divisible unital residuated binar** is a residuated binar $(A, \le, \cdot, 1, /)$ such that $x/x = 1$, $1x = x$ and $(y/x)x = (x/y)y$ hold

The partial order is definable by $x \le y \iff x = (x/y)y$

Note that $(x/y)y$ is a lower bound for any pair of elements x, y and we always have $1 \le 1/x$.

Theorem

In a right-divisible unital binar the partial order is down-directed and the identity $1/x = 1$ holds. The order is also definable by $x \le y \iff y/x = 1$.

Right-divisible unital residuated binars

Theorem

$(A, \cdot, 1, /)$ is a right-divisible unital residuated binar if and only if it satisfies the (quasi)identities $\quad x/x = 1 \qquad 1x = x$

$(y/x)x = (x/y)y$

$x/y = 1$ *and* $y/z = 1 \implies x/z = 1$

$z/xy = 1 \iff (z/y)/x = 1$

Note: $x \leq y$ if and only if $y/x = 1$. This is a partial order:

- **reflexive** by $x/x = 1$
- **antisymmetric** since if $x/y = 1$ and $y/x = 1$ then $x = 1x = (y/x)x = (x/y)y = 1y = y$
- **transitive** by the implication above

Open problem: Can the quasiequations be replaced by **identities**?

Open problem: Is the (quasi)equational theory **decidable**?

The right hoop identity

Adding one more identity produces an interesting sub**variety**

In the arithmetic of real numbers (or in any field) the following equation is fundamental to the **simplification of nested fractions**:

$$\frac{\frac{x}{y}}{z} = \frac{1}{z} \cdot \frac{x}{y} = \frac{x}{zy}$$

In a right-residuated binar this is the ***right hoop identity***:

$$(x/y)/z = x/zy$$

Consequences of the right hoop identity

Theorem

In a right divisible unital residuated binar the right hoop identity $x/yz = (x/z)/y$ *implies* $x(yz) = (xy)z$, $x1 = x$ *and* $x/1 = x$.

Proof.

$x(yz) = 1(x(yz))$ (left unital)
$= [(xy)z/(xy)z](x(yz))$ since $1 = x/x$
$= [((xy)z/z)/xy](x(yz))$ (right hoop id.)
$= [(((xy)z/z)/y)/x](x(yz))$ (right hoop id.)
$= [((xy)z/yz)/x](x(yz))$ (right hoop id.)
$= [(xy)z/x(yz)](x(yz))$ (right hoop id.)
$= [x(yz)/(xy)z]((xy)z)$ since $(y/x)x = (x/y)y$
$=$ reverse steps to get $= (xy)z$.
Now $x \leq 1$ implies $x = (x/1)1$, hence
$x1 = ((x/1)1)1 = (x/1)(11) = (x/1)1 = x$.
Finally $x/1 = (x/1)1 = (1/x)x = 1x = x$. □

Right generalized hoops

Definitions

A ***right generalized hoop*** $(A, \cdot, 1, /)$ is defined by the identities

$x/x = 1, \quad 1x = x, \quad (x/y)y = (y/x)x \quad$ and $\quad x/(yz) = (x/z)/y$

Define the **term-operation** $x \wedge y = (x/y)y$ and

a **binary relation** $\leq$ by $x \leq y \iff x = x \wedge y$

The next theorem shows that $\wedge$ is a **semilattice** operation

hence $\leq$ is a **partial order** on A

Moreover, A is **right-residuated** with respect to this order

and the left-unit 1 is the **top element**

Properties of right generalized hoops

Theorem

Let A be a right generalized hoop. Then
(i) *the term* $x \wedge y = (x/y)y$ *is idempotent, commutative and associative,*
(ii) $\leq$ *is a partial order and* $\wedge$ *is a meet operation with respect to* $\leq$*,*
(iii) $x \leq y \iff y/x = 1$ *for all* $x, y \in A$*,*
(iv) $xy \leq z \iff x \leq z/y$ *for all* $x, y, z \in A$*, and*
(v) $x \leq 1$ *for all* $x \in A$*, i.e., A is integral.*

Proof.

(i) The idempotence follows from the first two identities, and commutativity follows from the third.For associativity we calculate
$(x \wedge y) \wedge z = (((x/y)y)/z)z = (z/(x/y)y)(x/y)y$
$= ((z/y)/(x/y))(x/y)y$ (right hoop id.)
$= ((x/y)/(z/y))(z/y)y$ by assoc. and right-div
$= (x/(z/y)y)(z/y)y$ (right hoop id.)
$= x \wedge (z \wedge y) = x \wedge (y \wedge z)$ □

Proof.

(ii) Reflexivity, antisymmetry and transitivity of $\leq$ and the observation that $x \wedge y$ is the greatest lower bound of x, y follow from (i).

(iii) $x \leq y$ is equivalent to $x = (x/y)y$ hence $y/x = y/((x/y)y) = y/((y/x)x) = (y/x)/(y/x) = 1$, where the third equality uses the right hoop identity. Conversely, if $y/x = 1$ then $(x/y)y = (y/x)x = 1x = x$ and we conclude $x \leq y$.

(iv) From $xy \leq z$ we deduce $z/xy = 1$ by (iii). Hence $(x/(z/y))(z/y) = ((z/y)/x)x = (z/xy)x = 1x = x$, or equivalently $x \leq z/y$. Conversely, if $x \leq z/y$ then $x = (x/(z/y))(z/y) = (z/xy)x$, so $xy = (z/xy)xy = (xy/z)z$ which is equivalent to $xy \leq z$.

(v) Since $xy \leq xy$, (iv) implies $x \leq xy/y$. Multiplying by y gives $xy \leq (xy/y)y$, and the reverse inequality also holds by (iv). Hence $xy = (xy/y)y$, or equivalently $xy \leq y$. A final application of (iv) produces $x \leq y/y = 1$. □

Right generalized hoops and porrims

There is a 4-element right generalized hoop s.t. $\cdot$ is **not order-preserving in the right argument**

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	b	a
b	0	a	b	b
1	0	a	b	1

/	0	a	b	1
0	1	0	0	0
a	1	1	0	a
b	1	0	1	b
1	1	1	1	1

Partially ordered left-residuated integral monoids (or polrims for short) are left-residuated monoids such that the monoid operation is order-preserving in both arguments

They have been studied by **van Alten [1998]** and **Blok, Raftery [1997]**

Results on **polrims** do not automatically apply to right generalized hoops

Generalized hoops

Definition

A ***generalized hoop*** is an algebra $(A, \cdot, 1, \backslash, /)$ such that

- $(A, \cdot, 1, /)$ is a right generalized hoop, $(A, \cdot, 1, \backslash)$ is a left generalized hoop (defined by the mirror-image identities)
- and both these algebras have the same meet operation, i. e., the identity $(x/y)y = y(y\backslash x)$ holds

Generalized hoops were first studied by **Bosbach [1969]**

The name ***hoop*** was introduced by **Büchi and Owen [1975]**

Generalized hoops are also called ***pseudo hoops***

By the preceding theorem, they are **left- and right-residuated**

They are polrims, hence **congruence distributive** (**van Alten [1998]**)

Multiplication distributes over $\wedge$

In a residuated binar, the residuation property implies that $\cdot$ distributes over any existing joins in each argument. However, this is not true for meets.The following result was proved by **N. Galatos** for **GBL-algebras** but already holds for **generalized hoops**.

Theorem

In any generalized hoop $(x \wedge y)z = xz \wedge yz$ and $x(y \wedge z) = xy \wedge xz$.

Proof.

From $xz \leq xz$ it follows that $x \leq xz/z$, hence $xz \leq (xz/z)z$.Likewise, from $xz/z \leq xz/z$ we deduce $(xz/z)z \leq xz$, therefore $xz = (xz/z)z$.Note that $(x \wedge y)z \leq xz \wedge yz$ always holds since $\cdot$ is order-preserving.
Now $xz \wedge yz = (xz/yz)yz = ((xz/z)/y)yz$ (right hoop id.)
$= (y/((xz)/z))(xz/z)z$ by assoc. and divisibility
$= (y/((xz)/z))xz$ by the derived identity
$\leq (y/x)xz = (y \wedge x)z$ since $x \leq (xz)/z$. The second identity is similar. □

Not true for right generalized hoops

In the last step we made use of the implication $x \leq y \Rightarrow z/y \leq z/x$ which holds in all residuated binars.

The preceding result requires that $\cdot$ is **order-preserving in the right argument**

Recall the 4-element right generalized hoop from earlier

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	b	a
b	0	a	b	b
1	0	a	b	1

/	0	a	b	1
0	1	0	0	0
a	1	1	0	a
b	1	0	1	b
1	1	1	1	1

$(a \wedge b)a = (a/b)ba = 0ba = 0$ while $aa \wedge ba = a \wedge a = (a/a)a = 1a = a$.

From generalized hoops to GBL-algebras

Generalized hoops have a **simple** equational definition using only $\cdot, 1, \backslash, /$

Generalized hoops have a $\wedge$-**operation**, but **no join** (in general)

However, every finite generalized hoops is a **reduct of an integral GBL-algebra** since a finite meet semilattice with a top element is a lattice

Moreover, **finite** GBL-algebras are **commutative [J. - Montagna 2006]**

Definition

A ***hoop*** is a commutative generalized hoop, i.e. $xy = yx$.

Hence **finite** generalized hoops are in fact **hoops**

Open problem: Is the equational theory of generalized hoops **decidable**?

Homomorphic images of residuated lattices

An **efficient** way to construct some **homomorphic images**:

A **congruence relation** θ of a residuated lattice is determined by the **congruence class** $[1]_\theta$

Assume this congruence class has a **smallest element** c

Such an element is a **negative central idempotent**

i.e., $c \leq 1$, $cc = c$ and $cx = xc$ for all x

Let $I_A = \{c \in A : cc = c \leq 1 \text{ and } (\forall x \in A)\, cx = xc\}$

$(I_A, \cdot, \vee)$ is **dually isomorphic** to the **congruence lattice** of any **finite** A

Define $Ac = \{xc : x \in A\}$, and operations $u \wedge_c v = (u \wedge v)c$,

$$u/_c v = (u/v)c \quad u\backslash_c v = (u\backslash v)c$$

Theorem

Let A be a residuated lattice and $c \in I_A$. Then $A_c = (Ac, \wedge_c, \vee, \cdot, c, \backslash_c, /_c)$ is a residuated lattice and the map $h : A \to Ac$ given by $h(x) = xc$ is a surjective homomorphism onto A_c.
If θ is the kernel of h then xc is the smallest element of $[x]_\theta$.

Proof.

Ac is closed under the operations: $xc \vee yc = (x \vee y)c$ and $(xc)(yc) = xyc$ are both in Ac, and for $\wedge_c, /_c, \backslash_c$ this holds by construction.
h is surjective, so check that it is a homomorphism, then the homomorphic image is a residuated lattice (homomorphisms preserve identities)
Distributivity of $\cdot$ over $\vee$ shows that h preserves $\vee$
Centrality, idempotence and associativity imply that h preserves $\cdot$
$h(x) \wedge_c h(y) = (xc \wedge_c yc) = (xc \wedge yc)c \leq (x \wedge y)c = h(x \wedge y)$ since $c \leq 1$,
$(x \wedge y)c \leq xc$ and $(x \wedge y)c \leq yc$ imply $(x \wedge y)c \leq (xc \wedge yc)c$ □

Proof.

From $(x/y)y \le x$ we get $(x/y)yz \le xz$ and hence $x/y \le xz/yz$.
In particular, $(x/y)c \le (xc/yc)c$, which proves $h(x/y) \le h(x)/_c h(y)$.
For the opposite inequality we have $(xc/yc)yc \le xc \le x$, hence by centrality and idempotence $(xc/yc)c \le (x/y)c$.
Finally, $h(1) = 1c = c$, which is the unit of A_c. □

The theorem works for arbitrary residuated lattices. However in general it does not construct all homomorphic images, only those where the 1-congruence class of the kernel (and hence every congruence class) has a smallest element.

Corollary

Let A be a finite (or complete) residuated lattice and B any (complete) homomorphic image of A. Then B is isomorphic to A_c where c is the smallest negative central idempotent of A that is mapped by the homomorphism to 1 in B.

Hom. images of GBL-algebras and generalized hoops

Theorem

Let A be a GBL-algebra with a central idempotent element $c \in A$. Then A_c is isomorphic to the ***principal ideal*** *$\downarrow c$, hence $\wedge_c = \wedge$ and the map $h(x) = xc$ does not identify any elements of this ideal.*

Proof.

By **divisibility**, if $x \leq c$ then $x = (x/c)c$, and therefore $x \in A_c$.
Also, $h(x) = xc = (x/c)cc = x$, so $h \restriction_{\downarrow c}$ is the identity map. □

These results also apply to generalized hoops, and in the finite setting describe all homomorphic images

A version for right-generalized hoops is under investigation

Residuated structures for other logics

Residuated lattices have been studied since 1938 by **Dilworth** and **Ward** as abstractions of **ideal lattices of rings**

Boolean algebras with (residuated) operators were studied by **Jónsson** and **Tarski** 1951/2

- In logic **implication** plays the role of **residual**
- (generalized) **conjunction** plays the role of **multiplication**

Since the 1980s **residuated lattices** have been viewed as algebraic semantics of **substructural logics**

In computer science **bunched implication logic** was introduced in 1998

with (commutative) **residuated Heyting algebras** as algebraic semantics

Lattices with operators and some subclasses

LO	$\supseteq$	Residuated lattices
DLO	$\supseteq$	ℓ-groups GBL-algebras
HAO	$\supseteq$	Generalized bunched implication algebras
BAO	$\supseteq$	Residuated Boolean monoids

Generalized bunched implication algebras

Recall that a **Heyting algebra** is a residuated lattice with $\bot$ as bottom element and $xy = x \wedge y$

In this case we write $x \to y$ instead of $x\backslash y \quad (= y/x)$

A **generalized bunched implication algebra** or **GBI-algebra** is an algebra $(A, \vee, \wedge, \to, \bot, \cdot, 1, \backslash, /)$ where $(A, \vee, \wedge, \to, \bot)$ is a **Heyting algebra**, and $(A, \vee, \wedge, \cdot, 1, \backslash, /)$ is a **residuated lattice**

Theorem (Galatos - J. 2016)

*The equational theory of GBI-algebras is **decidable***
Also true for non-associative GBI-algebras, and for any subvarieties defined by finitely many "simple" identities using only $\wedge, \vee, \cdot, 1, \top$

However **divisibility** is **not equivalent** to any such identities

Residuated Boolean monoids

BI-algebras are **commutative** GBI-algebras; also equationally **decidable**

Applications in computer science; basis of **separation logic**

If **Heyting algebra** is replaced by **Boolean algebra**, we get **classical GBI-algebras**, also known as **residuated Boolean monoids**.

Theorem (Kurucz et. al. 1995)

The equational theory of residuated Boolean monoids is ***undecidable*** *(and also for many subvarieties, including classical BI-algebras)*

Moreover, **homomorphic images** of **finite GBI-algebras** can be computed as for residuated lattices

They are **determined** by $c \in I_A$ with the property that $\downarrow(\top c) \subseteq Ac$

(since Heyting algebras are GBL-algebras)

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Thank you