Fragments of intuitionistic logic and proof complexity

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Let L be a propositional logic (i.e., a Tarski-style structural consequence relation) in a propositional language C_0 (i.e., a set of connectives of specified finite arities), and L_{C_1} its fragment in a sublanguage $C_1 \subseteq C_0$. Clearly, if L' is a logic extending L, then its fragment L'_{C_1} is a logic extending L_{C_1}; that is, fragments of extensions are extensions of fragments. However, conversely, an extension of a fragment of L is *not* necessarily a fragment of an extension of L, and we are interested in conditions when this happens.

Let us stick to axiomatic extensions for simplicity. An axiomatic extension of the fragment L_{C_1} can be written as $L_{C_1} + X$ for some set X of C_1-formulas. Clearly, if this is the C_1-fragment of any extension of L, the least such extension is $L + X$. Thus, we can reformulate the problem as follows: is it true that $L_{C_1} + X = (L + X)_{C_1}$ for all sets X of C_1-formulas? (For finitary logics L, it is enough to consider finite X.) Generalizing the setup a little bit, we arrive at the following concept (the property discussed above is called hereditary C_1-conservativity of L over L_{C_1} under the definition below).

Definition 1. Let C_0 and C_1 be languages with a common sublanguage C, and L_i a logic in language C_i for $i = 0, 1$. We say that L_0 *is hereditarily C-conservative over* L_1 if $(L_0 + X)_C \subseteq (L_1 + X)_C$ for all sets X of C-formulas.

We are particularly interested in fragments of intuitionistic logic **IPC** containing implication. Let $C_{\mathbf{IPC}} = \langle \to, \wedge, \vee, \bot \rangle$ denote the language of **IPC**. The following characterization is due to Wroński [2].

Theorem 2. *Let $C_0, C_1 \subseteq C_{\mathbf{IPC}}$ and $\to \in C \subseteq C_0 \cap C_1$. The following are equivalent:*

1. $\mathbf{IPC}_{C_0}$ *is hereditarily C-conservative over* $\mathbf{IPC}_{C_1}$.

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2. $\wedge \in C_1$ *or* $C_0 \subseteq C_1$. □

It is perhaps more illuminative to split the result to the following cases:

- If $\{\to, \wedge\} \subseteq C \subseteq C_{\mathbf{IPC}}$, then **IPC** is hereditarily C-conservative over $\mathbf{IPC}_C$;
- $\mathbf{IPC}_{\to,\wedge}$ is not hereditarily $\to$-conservative over $\mathbf{IPC}_{\to,\vee,\bot}$;
- $\mathbf{IPC}_{\to,\bot}$ is not hereditarily $\to$-conservative over $\mathbf{IPC}_{\to,\vee}$;
- $\mathbf{IPC}_{\to,\vee}$ is not hereditarily $\to$-conservative over $\mathbf{IPC}_{\to,\bot}$.

It turns out Theorem 2 has close connections to certain issues in proof complexity of **IPC** and other superintuitionistic (si) logics.

If L is a (finitely axiomatizable) si logic or its fragment, let L-*EF* denote the *extended Frege* system for L: a proof system where formulas are derived by successive applications of instances of finitely many schematic axioms and rules, and also allowing for abbreviations of formulas by new variables (or alternatively, operating with circuits rather than just formulas).

If L is an si logic, we may assume wlog that L-*EF* is axiomatized by a standard axiom system for **IPC** together with an additional axiom α, called the *proper axiom* of the system. We can always take α $\wedge$-free; depending on properties of L, we may also take it $\vee$-free or $\bot$-free (in particular, L has an implicational proper axiom iff it is a subframe logic, and it has a $\{\to, \bot\}$-axiom iff it is a cofinal-subframe logic).

The positive part of Theorem 2 has an effective counterpart on the level of proofs [1]:

Theorem 3. *Let* $\{\to, \wedge\} \subseteq C \subseteq C_{\mathbf{IPC}}$*, and* L *be an si logic whose proper axiom is a* C*-formula. Given an* L*-EF proof of a* C*-formula* φ*, we can construct in polynomial time an* L_C*-EF proof of* φ*. That is,* L_C*-EF p-simulates* L*-EF on* C*-formulas.* □

Another way to state the theorem is that given an L-*EF* proof of φ, we can efficiently eliminate $\vee$ from the proof as long as it appears neither in the proper axiom of L nor in φ (and similarly for $\bot$). A more general result also holds where we allow φ arbitrary, and eliminate all disjunctions from the proof except for subformulas of φ.

Unlike $\vee$ and $\bot$, the construction in Theorem 3 does not eliminate $\wedge$; in fact, it may even introduce conjunctions that were not in the original proof. This difficulty with $\wedge$ is explained by the negative part of Theorem 2: a strong form of elimination of $\wedge$ is just false—if α is, say, an implicational axiom, we cannot in general eliminate conjunctions from $(\mathbf{IPC} + \alpha)$-*EF* proofs of implicational formulas *at all*, let alone efficiently.

We can partially remedy the situation by imposing additional restrictions. In particular, we can prove the following conjunction elimination result which is good enough for **IPC** itself:

Theorem 4. *Let α be an implicational axiom such that $(\mathbf{IPC}+\alpha)_\to = \mathbf{IPC}_\to+\alpha$. Given an $(\mathbf{IPC} + \alpha)$-EF proof of a formula φ, we can construct in polynomial time a proof of φ that contains no $\land$, $\lor$, or $\bot$ apart from subformulas of φ.*

A similar statement holds for $\{\to, \bot\}$-axioms α. $\Box$

Question 5. *Can we efficiently eliminate $\land$ from proofs in si logics whose proper axioms involve disjunctions, under reasonable conditions?*

References

[1] Emil Jeřábek, *Proof complexity of intuitionistic implicational formulas*, preprint, 2015, 45 pp., arXiv:1512.05667 [cs.LO].

[2] Andrzej Wroński, *On reducts of intermediate logics*, Bulletin of the Section of Logic 9 (1980), no. 4, pp. 176–179.