Counting Tolerances on Finite Lattices

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Content

1. Tolerances for lattices

2. Combinatorial playground

Definition

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The transitive closure of a tolerance relation is a lattice congruence.

Every lattice congruence is a lattice tolerance and its blocks are congruence classes.

The lattice of tolerances

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The lattice of all congruences on L, denoted by $\mathrm{Con}(L)$, is a subposet of $\mathrm{Tol}(L)$, but it is not necessarily its sublattice.

The lattice of blocks of a tolerance

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If α and β are blocks of T, then $\{a \vee b \mid a \in \alpha, b \in \beta\}$ and $\{a \wedge b \mid a \in \alpha, b \in \beta\}$ are preblocks of T. Blocks containing these preblocks are uniquely determined and they are, respectively, the join and the meet of blocks α, β in the lattice of all blocks of T.

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This lattice, denoted by L/T, is called the factor lattice of L modulo T.

Blocks of finite lattices

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Therefore, if α is a block of T, then we use the notation $\alpha = [0_\alpha, 1_\alpha]$. It means that any tolerance T on L can be represented by the system of its blocks.

Characterisation

Lemma (G. Czédli, 1982)

For a finite lattice L, let $\mathcal{B}$ be a collection of nonempty subsets of L. Then $\mathcal{B}$ is the set of all blocks of some tolerance of L iff $\mathcal{B}$ is of the form $\{[a_\gamma, b_\gamma] \colon \gamma \in \Gamma\}$, where $[a_\gamma, b_\gamma]$ are intervals of L and the following conditions are satisfied:

(i) $\bigcup_{\gamma \in \Gamma} [a_\gamma, b_\gamma] = L$;

(ii) *for any* $\gamma, \delta \in \Gamma$, $a_\gamma = a_\delta$ *is equivalent to* $b_\gamma = b_\delta$;

(iii) *for any* $\gamma, \delta \in \Gamma$, *there exists* $\mu \in \Gamma$ *such that* $a_\mu = a_\gamma \vee a_\delta$ *and* $b_\mu \geq b_\gamma \vee b_\delta$.

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The (unique) smallest glued tolerance of L is called the skeleton tolerance of L, and it is denoted by $\Sigma(L)$.

Glued tolerances on chains

Lemma

A collection $\mathcal{B}$ of subsets of the chain $\mathcal{C}_n = \langle\{0, \dots, n-1\}, \leq\rangle$ is the set of all blocks of some glued tolerance of L iff $\mathcal{B}$ is of the form

$$\{[n_i, m_i] \colon i = 1, \dots, k\}$$

for some $1 \leq k \leq n-1$, where $n_1 = 0$, $m_k = n-1$ and $n_i < n_{i+1} \leq m_i < m_{i+1}$ for all $i = 1, \dots, k$.

Number of congruences

Theorem

For every $n > 1$ and every lattice L with n elements,

$$2 \leq |\operatorname{Con}(L)| \leq |\operatorname{Con}(\mathcal{C}_n)| = 2^{n-1}.$$

Tolerances on chains

Theorem (J. Grygiel and S. Radeleczki, 2013)

For every finite lattice L and every $T \in \mathrm{Tol}(L)$ we have

$$\mathrm{Tol}(L/T) \cong [T)_{\sqsubseteq},$$

where $[T)_{\sqsubseteq} = \{S \in \mathrm{Tol}(L) \colon T \sqsubseteq S\}$.

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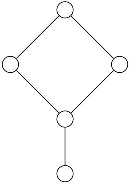
Theorem

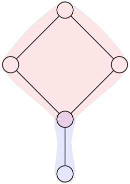
For every $n \geq 1$, *we have* $\mathrm{Glu}(\mathcal{C}_{n+1}) \cong \mathrm{Tol}(\mathcal{C}_{n+1}/\Sigma) \cong \mathrm{Tol}(\mathcal{C}_n)$.

Thus, $\mathrm{Tol}(\mathcal{C}_n)$ is isomorphic to a principal filter of $\mathrm{Tol}(\mathcal{C}_{n+1})$.

Not always $\mathrm{Glu}(L) \cong \mathrm{Tol}(L/\Sigma)$

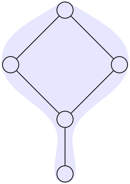


L

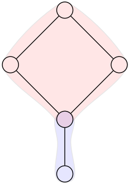


L/Σ

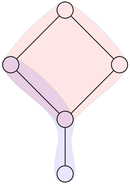
$\mathrm{Tol}(L/\Sigma)$



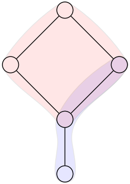
T_1



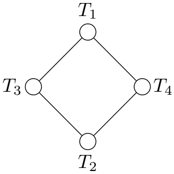
T_2



T_3



T_4



$\mathrm{Glu}(L)$

Dyck words

A Dyck word of length $2n$ is a word consisting of n letters u and n letters d such that no initial segment of the string has more u's than d's.

Dyck words of size $2n$ are enumerated by Catalan numbers $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$.

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There exists a bijection between the set of glued tolerances on $\mathcal{C}_n$ and the set of Dyck words of length $2n-2$.

Encoding

Let T be a glued tolerance on $\mathcal{C}_n$ and let $\mathcal{B}=\{[n_i, m_i] \colon i = 1, \ldots, k\}$ be the set of its blocks. Let $n_0 := 0$ and $n_{k+1} := n - 1$.

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Given T, we construct the corresponding Dyck word as follows:

for $i = 1, \ldots, k$:
  add $m_i - m_{i-1}$ letters u,
  add $n_{i+1} - n_i$ letters d.

Decoding

Let s be a Dyck word of length $2n-2$. We divide s into subwords starting with u and ending with d. Suppose that s consists of k such subwords:

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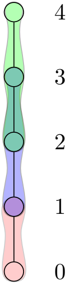
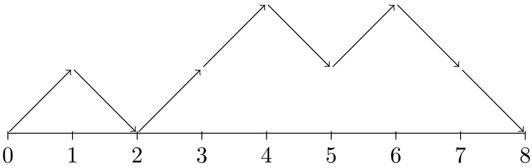
We define blocks $[n_i, m_i]$ for $i = 1, \dots, k$ as follows:

$$\begin{aligned} n_i &:= m_{i-1} - u(s_1 \dots s_{i-1}) + d(s_1 \dots s_{i-1}), \\ m_i &:= m_{i-1} + u(s_i). \end{aligned}$$

Example

For the tolerance on L_5 with three blocks $s_1 = [0, 1], s_2 = [1, 3], s_3 = [2, 4]$ the corresponding Dyck word has the form *ud uud udd*.





Number of tolerances

Theorem

For every $n \geq 1$, we have

$$|\operatorname{Tol}(\mathcal{C}_n)| = |\operatorname{Glu}(\mathcal{C}_{n+1})| = \operatorname{Cat}_n = \frac{1}{n+1}\binom{2n}{n}.$$

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Corollary

Asymptotically, every fourth tolerance on a chain is glued.

Tolerances with a prescribed number of blocks

Let $T(z,u) = \sum_{n,k=0}^{\infty} T(n,k) z^n u^k$ be a bivariate generating function enumerating all tolerances of a given size (marked by z) and with a given number of blocks (marked by u).

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$$\begin{aligned} T(z,u) &= 1 + zuT(z,u) + z^2uT^2(z,u) + z^3uT^3(z,u) + \dots \\ &= 1 + \frac{zuT(z,u)}{1 - zT(z,u)}, \end{aligned}$$

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which leads to the solution in the form

$$T(z,u) = \frac{1 - z - uz - \sqrt{(uz - z - 1)^2 - 4z}}{2z}.$$

Tolerances with a prescribed number of blocks

The number of tolerances on the chain of size n with exactly k blocks:

$n\backslash k$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	1			
4	1	6	6	1		
5	1	10	20	10	1	
6	1	15	50	50	15	1

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These are the so-called Narayana numbers:

$$[z^n u^k] T(z,u) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Expected number of blocks

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The expected number of blocks in a tolerance on the n-element chain is equal to

$$\frac{[z^n]B(z)}{[z^n]T(z)} = \frac{\binom{2n-1}{n}}{\frac{1}{n+1}\binom{2n}{n}} = \frac{n+1}{2}.$$

Tolerances with bounded degree

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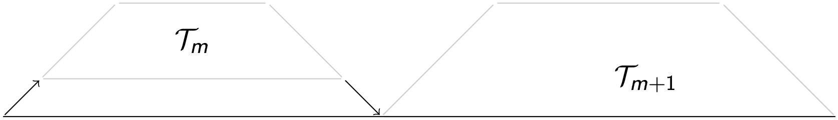
For technical reasons we assume that there exists an empty tolerance, which is of any degree.

It is clear that every congruence is a tolerance of degree 0.

Tolerances with bounded degree

For every $m \in \mathbb{N}$, let $T_m(z)$ denote the generating function for the sequence enumerating all tolerances of degree m. Then

$$T_0(z) = \frac{1-z}{1-2z},$$
$$T_{m+1}(z) = \frac{1}{1 - zT_m(z)}.$$



Decomposition of a non-empty tolerance of degree $m+1$

Tolerances with bounded degree

First few functions along with inverses of their dominant singularities:

$$
\begin{aligned}
T_0(z) &= \frac{1-z}{1-2z} && 2 \\
T_1(z) &= \frac{1-2z}{1-3z+z^2} && 2.618 \\
T_2(z) &= \frac{1-3z+z^2}{1-4z+3z^2} && 3 \\
T_3(z) &= \frac{1-4z+3z^2}{1-5z+6z^2-z^3} && 3.2469
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The sequence of singularities tends to 4.

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For every $m \in \mathbb{N}$ the generating function T_m is rational, while the limit generating function is algebraic yet not rational.

Tolerances with bounded degree

First few sequences:

$$[z^n]T_0(z) = 2^n$$

$$[z^n]T_1(z) = \sum_{k=0}^{n-1} \binom{n+k-1}{2k} = F_{2n}$$

$$[z^n]T_2(z) = \frac{3^{n-1}+1}{2}$$

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Ultimate Question of Life, the Universe, and Everything

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42

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$$\mathrm{Cat}_5 = 42$$

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Generalization:

Ultimate Question of Life, the Universe, and Everything

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Generalization: Catalan


THE END