Counting Tolerances on Finite Lattices

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Abstract

Our goal is to estimate the number of tolerances in finite lattices. In particular, we focus on enumerating all congruences and tolerances on finite chains and show the correspondence between these relations and Dyck paths. Moreover, we study specific kinds of tolerances and provide some statistical properties about random ones.

1 Introduction

A *tolerance relation* on a lattice L is a reflexive and symmetric relation compatible with the operations of L. Let us denote the set of all tolerances on L by $\mathrm{Tol}(L)$. It is clear that every congruence of a lattice L is a tolerance on L. The tolerances of a lattice L, ordered by inclusion, form an algebraic lattice denoted by $\mathrm{Tol}(L)$ (see [2]). The lattice of all congruences on L, which will be denoted by $\mathrm{Con}(L)$, is a subposet of $\mathrm{Tol}(L)$, but it is not necessarily its sublattice.

Let $T \in \mathrm{Tol}(L)$. The maximal subset X of L such that every two element from X are in the relation T is called a *block* of T. Blocks of T are convex sublattices by [1] and [2] and they form a lattice called *the factor lattice of L modulo T* ([3]).

A tolerance T on a lattice L is called *glued* (see [6]) if its transitive closure is the total relation L^2. The sublattice of all glued tolerances of L will be denoted by $\mathrm{Glu}(L)$. The (unique) smallest glued tolerance of L is called the *skeleton tolerance* of L, and it is denoted by $\Sigma(L)$. A tolerance is of *degree* $m \in \mathbb{N}$ if every two blocks share at most m elements. By definition, if a tolerance is of degree m, then it is of degree $m+1$. It is clear that every congruence is a tolerance of degree 0.

From now on we will deal only with finite lattices. In this case blocks of a tolerance $T \in \mathrm{Tol}(L)$ are intervals of L. It means that any tolerance $T \in \mathrm{Tol}(L)$ can be represented by the system of its blocks.

2 Main results

Our main results concentrate on quantitative aspects of tolerances on lattices, with a special emphasis on enumerating congruencese as well as tolerances and their blocks in the case of finite chains. In what follows, let us denote the n-element chain by $\mathcal{C}_n$.

First of all, by means of the notion of c-perspectivity (see [4]) and some known algebraic facts, we prove that for every $n > 1$ the number of congruences on $\mathcal{C}_n$ determines the upper limit for the number of congruences for any n-element lattice, i.e.,

Theorem 1. *For every $n > 1$ and every lattice L with n elements,*

$$2 \leq |\operatorname{Con}(L)| \leq |\operatorname{Con}(\mathcal{C}_n)| = 2^{n-1}.$$

Although, in general, it is not true that $\operatorname{Glu}(L) \cong \operatorname{Tol}(L/\Sigma)$, by applying some results from [5], we can prove the following result, which implies that $\operatorname{Tol}(\mathcal{C}_n)$ is isomorphic to a principal filter of $\operatorname{Tol}(\mathcal{C}_{n+1})$.

Theorem 2. *For every $n \geq 1$, we have $\operatorname{Glu}(\mathcal{C}_{n+1}) \cong \operatorname{Tol}(\mathcal{C}_n)$.*

Next, we construct a recursive bijection between all tolerances on the finite chain $\mathcal{C}_n$ and all Dyck paths (see [7]) of length $2n$, which are enumerated by Catalan numbers (denoted here by Cat_n). This bijection, along with Theorem 2, leads us to the following quantitative outcome.

Theorem 3. *For every $n \geq 1$, we have*

$$|\operatorname{Tol}(\mathcal{C}_n)| = |\operatorname{Glu}(\mathcal{C}_{n+1})| = \mathsf{Cat}_n = \frac{1}{n+1}\binom{2n}{n}.$$

Using the theory of bivariate generating functions, we can enumerate tolerances with a prescribed number of blocks.

Theorem 4. *For every $n > 1$, the number of all tolerances on $\mathcal{C}_n$ consisting of exactly $k \geq 1$ blocks is given by the formula*

$$T(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Moreover, the expected number of blocks in a tolerance on $\mathcal{C}_n$ is equal to $\frac{n+1}{2}$.

By means of symbolic methods in enumerative combinatorics, we are able to recursively enumerate tolerances of any degree. Only for technical reasons, let us assume the existence of an empty chain with one empty tolerance. The following result constitutes a generalization of Theorems 1 and 3.

Theorem 5. *For every $m \in \mathbb{N}$, let $T_m(z)$ denote the generating function for the sequence enumerating all tolerances of degree m on a finite chain. Then*

$$T_0(z) = \frac{1-z}{1-2z},$$
$$T_{m+1}(z) = \frac{1}{1-zT_m(z)}.$$

For example, on the chain $\mathcal{C}_n$, where $n > 0$, there are $\sum_{k=0}^{n-1} \binom{n+k-1}{2k}$ tolerances of degree 1 and $\frac{3^{n-1}+1}{2}$ tolerances of degree 2.

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