Sheaf representations via duality

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Étale space

An étale space is a local homeomorphism $p\colon E \to X$, that is,

$$\forall e \in E \quad \exists\, V \subseteq E,\ U \subseteq X \text{ both open with}$$
$$e \in V \text{ and } p|V\colon V \to U \text{ is a homeomorphism}$$

A local section of p is a continuous $s\colon U \to E$ with $U \subseteq X$ open and $p \circ s = id_U$

A global section of p is a local section with $U = X$

For any $S \subseteq X$ let

$$\Gamma_p(S) = \{s\colon S \to E \mid s \text{ is continuous and } p \circ s = id_S\}$$

Sheaf as a functor

A presheaf of algebras in a variety $\mathcal{V}$ over a space X is a functor

$$F\colon (\Omega X)^{op} \to \mathcal{V}$$

F is a sheaf provided it satisfies the following patching property

For any collection of opens $(U_i)_{i \in I}$ of X and $(s_i)_{i \in I}$ with $s_i \in FU_i$ for each $i \in I$ with

$$\forall i, j \qquad s_i|(U_i \cap U_j) = s_j|(U_i \cap U_j),$$

then there exists a unique $s \in F(\bigcup_{i \in I} U_i)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

The categories of étale spaces over X and sheaves over X are equivalent

Soft sheaf representations

A sheaf with étale space $p\colon E \to Y$ is said to be soft provided for every compact saturated $K \subseteq Y$ and continuous section $s\colon K \to E$ of p, there exists a global section t of p such that $t|_K = s$.

Let A be an algebra in a variety $\mathcal{V}$. A soft sheaf representation of A is a soft sheaf of $\mathcal{V}$-algebras whose algebra of global sections is isomorphic to A.

Stone's representation theorem for Boolean algebras may be seen as soft sheaf representations in which all stalks are the two-element Boolean algebra.

Compact pospaces

$(Y, \tau, \leq)$ is a compact ordered space provided

- (Y, τ) is a compact space
- $\leq$ is a partial order on Y
- $\leq$ is a closed in $(Y, \tau) \times (Y, \tau)$.

It follows that Y is Hausdorff. In fact it satisfies the following ordered variant of Hausdorff

$$\forall x, y \quad (x \not\leq y \implies \exists U, V \text{ open}, U \text{ an up-set}, V \text{ a down-set with } x \in U,\ y \in V \text{ and } U \cap V = \emptyset)$$

Co-compact dual

Let Y be a compact pospace, $Y^{\uparrow} = (Y, \tau \cap Up(Y, \leq))$ and $Y^{\downarrow} = (Y, \tau \cap Down(Y, \leq))$ the associated stably compact spaces

Let $S \subseteq Y$. Then the following are equivalent:

(i) S is a closed up-set

(ii) S is a is compact saturated in $Y^{\uparrow}$

(iii) S is a closed in $Y^{\downarrow}$.

In particular, U is open in $Y^{\downarrow}$ if and only if U^c is compact saturated in $Y^{\uparrow}$. $Y^{\downarrow}$ is the co-compact dual of $Y^{\uparrow}$ (and vice versa)

Soft sheaf representations of algebras

Joint work with Sam van Gool

Theorem: There is a bijection between isomorphism classes of soft sheaf representations of A over $Y^{\uparrow}$ and frame homomorphisms $\Omega Y^{\downarrow} \to \mathit{Con}\, A$ into subframes of pairwise permuting congruences.

Corollary: Let $\mathcal{V}$ be congruence permutable, congruence distributive with CIP. For all A in $\mathcal{V}$, $\mathit{Con}\, A \cong \Omega Y^{\downarrow}$ for some Priestley space Y and A has a soft sheaf representation over $Y^{\uparrow}$.

Theorem: Let $f\colon Y_1 \to Y_2$ be a morphism of compact ordered spaces and $F_1\colon \Omega Y_1^{\uparrow} \to \mathcal{V}$ a soft sheaf representation of A with corresponding frame homomorphism $\psi_1\colon \Omega Y_1^{\downarrow} \to \mathit{Con}\, A$. Then $F_2 = F_1 \circ \Omega f^{\uparrow}\colon \Omega Y_2^{\uparrow} \to \mathcal{V}$ is a soft sheaf representation of A and the corresponding frame homomorphism is $\psi_2 = \psi_1 \circ \Omega f^{\downarrow}\colon \Omega Y_2^{\downarrow} \to \mathit{Con}\, A$.

Permuting DL congruences

Joint work with Sam van Gool

Lemma: Let A be a DL, X its Priestley spectrum. Let $\theta_1, \theta_2 \in Con\, A$ and C_1, $C_2 \subseteq X$ closed. The following are equivalent:

(i) θ_1 and θ_2 permute;

(ii) For every $x, x' \in X$ with one in C_1 and the other in C_2, if $x \leq x'$ then there exists $x'' \in C_1 \cap C_2$ with $x \leq x'' \leq x'$.

Definition: Let X be a Priestley space and Y a compact pospace. A continuous function $q \colon X \to Y^{\downarrow}$ is interpolating provided, for all $x, x' \in X$, if $x \leq x'$, then there exists $x'' \in X$ with $q(x), q(x') \leq q(x'')$ and $x \leq x'' \leq x'$.

Soft sheaf representations of DLEs

Joint work with Sam van Gool

Theorem: Let A be a distributive lattice, X its Priestley spectrum. There is a bijection between isomorphism classes of soft sheaf representations of A over $Y^{\uparrow}$ and interpolating decompositions of X over $Y^{\downarrow}$.

If A is a DLE then the soft sheaf representations of A correspond to those $q\colon X \to Y^{\downarrow}$ for which $X_y = q^{-1}(\uparrow y)$ is the dual of a DLE quotient of A for each $y \in Y$.

Theorem: Let A be a distributive lattice, X its Priestley spectrum, and $q\colon X \to Y_1^{\downarrow}$ interpolating. Let $f\colon Y_1 \to Y_2$ be a morphism of compact pospaces then $f^{\downarrow} \circ q\colon X \to Y_2^{\downarrow}$ is an interpolating decomposition of X over $Y_2^{\downarrow}$ and the corresponding soft sheaf representation over $Y_2^{\uparrow}$ is the direct-image sheaf given by $f^{\uparrow}$ of the soft sheaf representation over $Y_1^{\uparrow}$ corresponding to q.

Priestley decompositions of DLEs

Joint work with Anna Carla Russo and Peter Jipsen

Let A be a distributive lattice with Priestley spectrum X. Further let Y be a Priestley space. A Priestley decomposition of X over Y is a continuous surjection $q\colon X \to Y$ such that for all $x, x' \in X$

- $x \leq x' \implies f(x) \leq f(x')$;
- $f(x) < f(x') \implies x \leq x'$.

Theorem: There is a bijection between isomorphism classes of Priestley products of A over Y and Priestley decompositions of X over Y.

NB! Priestley products were introduced by Peter Jipsen and Franco Montagna for the study of (integral, commutative) GBL algebras.