The FEP for residuated lattices via local finiteness of the monoid reducts

Nick Galatos (joint work with R. Cardona)
University of Denver
ngalatos@du.edu

June, 2016

FEP and decidability

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

FEP and decidability

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

Fact. If $\mathcal{K}$ has the FEP and is finitely axiomatizable, then it's universal theory is decidable.

FEP and decidability

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

Fact. If $\mathcal{K}$ has the FEP and is finitely axiomatizable, then it's universal theory is decidable.

Fact. The decidability of the universal theory implies the decidability of the quasi-equational theory, which implies the decidability of the word problem, which implies the decidability of the equational theory.

FEP and decidability

A class of algebras $\mathcal{K}$ has the *finite embeddability property (FEP)* if for every $\mathbf{A} \in \mathcal{K}$, every finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ can be (partially) embedded in a finite $\mathbf{D} \in \mathcal{K}$.

Fact. If $\mathcal{K}$ has the FEP and is finitely axiomatizable, then it's universal theory is decidable.

Fact. The decidability of the universal theory implies the decidability of the quasi-equational theory, which implies the decidability of the word problem, which implies the decidability of the equational theory.

Fact. The FEP for a finitiely axiomatizable class $\mathcal{K}$ that forms the algebraic semantics of a finitary logical system $\vdash$, implies its *strong finite model property*:

if $\Phi \not\vdash \psi$, for finite Φ, then there is a finite counter-model.

Residuated lattices

A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

Residuated lattices

A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

Equations of the form $x^m \leq x^n$, for natural numbers m and n, are called *knotted equations*.

Residuated lattices

A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

Equations of the form $x^m \leq x^n$, for natural numbers m and n, are called *knotted equations*. We refer to the corresponding equality $x^m = x^n$ as *periodicity*.

Residuated lattices

A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

Equations of the form $x^m \leq x^n$, for natural numbers m and n, are called *knotted equations*. We refer to the corresponding equality $x^m = x^n$ as *periodicity*.

We consider the properties $xy = yx$ (commutativity), $x \leq 1$ (integrality) and $x \leq x^2$ (contraction).

FEP for RL

RL does **not** have the FEP.

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1) +$ (any eq'n over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL does **not** have the FEP.

RL $+ (x \leq 1) +$ (any eq'n over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

RL does **not** have the FEP.

RL $+ (x \leq 1) +$ (any eq'n over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

Most varieties RL $+ (x^m \leq x)$, do *not* have the FEP/dWP. (Horčík)

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1) +$ (any eq'n over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

Most varieties RL $+ (x^m \leq x)$, do *not* have the FEP/dWP. (Horčík)

RL $+ (x^m \leq x^n)$, for $n \neq 1$, does **not** have the FEP/dWP. (Horčík)

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1) +$ (any eq'n over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

Most varieties RL $+ (x^m \leq x)$, do *not* have the FEP/dWP. (Horčík)

RL $+ (x^m \leq x^n)$, for $n \neq 1$, does **not** have the FEP/dWP. (Horčík)

The variety RL $+ (yx = xy)$ does **not** have the FEP. (Blok, van Alten)

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1) +$ (any eq'n over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

Most varieties RL $+ (x^m \leq x)$, do *not* have the FEP/dWP. (Horčík)

RL $+ (x^m \leq x^n)$, for $n \neq 1$, does **not** have the FEP/dWP. (Horčík)

The variety RL $+ (yx = xy)$ does **not** have the FEP. (Blok, van Alten)

The varieties RL $+ (x^m \leq x^n) + (yx = xy)$ have the FEP. (van Alten)

RL does **not** have the FEP.

RL $+ (x \leq 1) +$ (any eq'n over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

Most varieties RL $+ (x^m \leq x)$, do *not* have the FEP/dWP. (Horčík)

RL $+ (x^m \leq x^n)$, for $n \neq 1$, does **not** have the FEP/dWP. (Horčík)

The variety RL $+ (yx = xy)$ does **not** have the FEP. (Blok, van Alten)

The varieties RL $+ (x^m \leq x^n) + (yx = xy)$ have the FEP. (van Alten)

The varieties RL $+ (x^m \leq x^n) + (xyx = xxy)^* +$ (any equation over $\{\vee, \cdot, 1\}$) has the FEP. (Cardona and G.)

FEP for RL

RL does **not** have the FEP.

RL $+ (x \leq 1)+$ (any eq'n over $\{\vee, \cdot, 1\}$) has the FEP. (G. and Jipsen)

RL $+ (x^2 \leq x)$ (mingle) has the FEP. (Horčík)

Most varieties RL $+ (x^m \leq x)$, do *not* have the FEP/dWP. (Horčík)

RL $+ (x^m \leq x^n)$, for $n \neq 1$, does **not** have the FEP/dWP. (Horčík)

The variety RL $+ (yx = xy)$ does **not** have the FEP. (Blok, van Alten)

The varieties RL $+ (x^m \leq x^n) + (yx = xy)$ have the FEP. (van Alten)

The varieties RL $+ (x^m \leq x^n) + (xyx = xxy)^* +$ (any equation over $\{\vee, \cdot, 1\}$) has the FEP. (Cardona and G.) The same holds more generally for equations:

$$xy_1xy_2 \cdots y_rx = x^{a_0}y_1x^{a_1}y_2 \cdots y_rx^{a_r}. \qquad (a)$$

Here $a = (a_0, a_1, \ldots, a_r)$ is a vector of natural numbers whose sum is $r+1$ (*balanced property*) and product is 0.

FEP via Residuated Frames

The construction in the above proof uses *residuated frames*.

FEP via Residuated Frames

The construction in the above proof uses *residuated frames*. The *submonoid* of A generated by B might not be finite neither its *powerset*, but $\mathbf{D}$ is based on a finite subset of that powerset. (The finiteness is established by using the theory of well-ordered sets.)

FEP via Residuated Frames

The construction in the above proof uses *residuated frames*. The *submonoid* of A generated by B might not be finite neither its *powerset*, but $\mathbf{D}$ is based on a finite subset of that powerset. (The finiteness is established by using the theory of well-ordered sets.) The construction of D naturally preserves all equations in the language $\{\vee, \cdot, 1\}$ that $\mathbf{A}$ satisfies. So, one could investigate FEP/undecidability questions for all varieties of residuated lattices axiomatized by $\{\vee, \cdot, 1\}$-identities.

FEP via Residuated Frames

The construction in the above proof uses *residuated frames*. The *submonoid* of A generated by B might not be finite neither its *powerset*, but $\mathbf{D}$ is based on a finite subset of that powerset. (The finiteness is established by using the theory of well-ordered sets.) The construction of D naturally preserves all equations in the language $\{\vee, \cdot, 1\}$ that $\mathbf{A}$ satisfies. So, one could investigate FEP/undecidability questions for all varieties of residuated lattices axiomatized by $\{\vee, \cdot, 1\}$-identities.

As mentioned above the algebra D might not be based on a subset of A, but in special cases (when the submoinoid of $\mathbf{A}$ generated by B is finite) this could happen and D could even be a subalgebra of $\mathbf{A}$ with respect to multiplication and join. The construction is essentially based on Tarski and McKinzie.

FEP via Residuated Frames

The construction in the above proof uses *residuated frames*. The *submonoid* of A generated by B might not be finite neither its *powerset*, but $\mathbf{D}$ is based on a finite subset of that powerset. (The finiteness is established by using the theory of well-ordered sets.) The construction of D naturally preserves all equations in the language $\{\vee, \cdot, 1\}$ that $\mathbf{A}$ satisfies. So, one could investigate FEP/undecidability questions for all varieties of residuated lattices axiomatized by $\{\vee, \cdot, 1\}$-identities.

As mentioned above the algebra D might not be based on a subset of A, but in special cases (when the submoinoid of $\mathbf{A}$ generated by B is finite) this could happen and D could even be a subalgebra of $\mathbf{A}$ with respect to multiplication and join. The construction is essentially based on Tarski and McKinzie.

This has applications in constructing *canonical formulas* for such varieties (joint work with N. Bezhanishvili and L. Spada), so we investigate further for which axiomatizations this holds.

FEP via Residuated Frames

For example we consider the case where the equation (a) is not balanced, or even more general simple equations.

FEP via Residuated Frames

For example we consider the case where the equation (a) is not balanced, or even more general simple equations.

Consider the equations:

$$xyxzxyx = yxzx^4yx$$
$$xxyxzxyx = xyxzx^4yx$$

We will show that the first one defines a variety of residuated lattices that has the FEP (hence a decidable universal theory) but the second one defines a variety with an undecidable word problem (hence a decidable universal theory).

The Tarski-McKinzei conucleus

Let $\mathbf{A}$ be a residuated lattice and B a (finite) subset of A such that the submonoid M of $\mathbf{A}$ generated by B is finite.

The Tarski-McKinzei conucleus

Let $\mathbf{A}$ be a residuated lattice and B a (finite) subset of A such that the submonoid M of $\mathbf{A}$ generated by B is finite. Then the join subsemilattice D of $\mathbf{A}$ generated by M is also finite (of size at most $2^{|M|}$).

The Tarski-McKinzei conucleus

Let $\mathbf{A}$ be a residuated lattice and B a (finite) subset of A such that the submonoid M of $\mathbf{A}$ generated by B is finite. Then the join subsemilattice D of $\mathbf{A}$ generated by M is also finite (of size at most $2^{|M|}$).

By the distributivity of multiplication over join, D is a submonoid

The Tarski-McKinzei conucleus

Let $\mathbf{A}$ be a residuated lattice and B a (finite) subset of A such that the submonoid M of $\mathbf{A}$ generated by B is finite. Then the join subsemilattice D of $\mathbf{A}$ generated by M is also finite (of size at most $2^{|M|}$).

By the distributivity of multiplication over join, D is a submonoid and of $\mathbf{A}$ and for every element x of $\mathbf{A}$ there is a largest element $\sigma(x)$ of M such that $\sigma(x) \leq x$.

The Tarski-McKinzei conucleus

Let $\mathbf{A}$ be a residuated lattice and B a (finite) subset of A such that the submonoid M of $\mathbf{A}$ generated by B is finite. Then the join subsemilattice D of $\mathbf{A}$ generated by M is also finite (of size at most $2^{|M|}$).

By the distributivity of multiplication over join, D is a submonoid and of $\mathbf{A}$ and for every element x of $\mathbf{A}$ there is a largest element $\sigma(x)$ of M such that $\sigma(x) \leq x$. Actually $\sigma(x)$ is the (finite) join of all of the elements of M which are below x.

The Tarski-McKinzei conucleus

Let $\mathbf{A}$ be a residuated lattice and B a (finite) subset of A such that the submonoid M of $\mathbf{A}$ generated by B is finite. Then the join subsemilattice D of $\mathbf{A}$ generated by M is also finite (of size at most $2^{|M|}$).

By the distributivity of multiplication over join, D is a submonoid and of $\mathbf{A}$ and for every element x of $\mathbf{A}$ there is a largest element $\sigma(x)$ of M such that $\sigma(x) \leq x$. Actually $\sigma(x)$ is the (finite) join of all of the elements of M which are below x.

Submonoids-subsemilattices with this maximum-existence property give rise to *conuclei* σ defined as above. Namely $\sigma : A \to A$ is an interior operator that further satisfies $\sigma(x) \cdot \sigma(y) \leq \sigma(xy)$ and $\sigma(1) = 1$.

The Tarski-McKinzei conucleus

Let $\mathbf{A}$ be a residuated lattice and B a (finite) subset of A such that the submonoid M of $\mathbf{A}$ generated by B is finite. Then the join subsemilattice D of $\mathbf{A}$ generated by M is also finite (of size at most $2^{|M|}$).

By the distributivity of multiplication over join, D is a submonoid and of $\mathbf{A}$ and for every element x of $\mathbf{A}$ there is a largest element $\sigma(x)$ of M such that $\sigma(x) \leq x$. Actually $\sigma(x)$ is the (finite) join of all of the elements of M which are below x.

Submonoids-subsemilattices with this maximum-existence property give rise to *conuclei* σ defined as above. Namely $\sigma : A \to A$ is an interior operator that further satisfies $\sigma(x) \cdot \sigma(y) \leq \sigma(xy)$ and $\sigma(1) = 1$. Then the structure $\mathbf{A}_\sigma = (\sigma[A], \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, 1)$ is a residuated lattice, where $\sigma[A] = D$, $x \wedge_\sigma y = \sigma(x \wedge y)$, $x \backslash_\sigma y = \sigma(x \backslash y)$ and $x /_\sigma y = \sigma(x/y)$.

The Tarski-McKinzei conucleus

Let $\mathbf{A}$ be a residuated lattice and B a (finite) subset of A such that the submonoid M of $\mathbf{A}$ generated by B is finite. Then the join subsemilattice D of $\mathbf{A}$ generated by M is also finite (of size at most $2^{|M|}$).

By the distributivity of multiplication over join, D is a submonoid and of $\mathbf{A}$ and for every element x of $\mathbf{A}$ there is a largest element $\sigma(x)$ of M such that $\sigma(x) \leq x$. Actually $\sigma(x)$ is the (finite) join of all of the elements of M which are below x.

Submonoids-subsemilattices with this maximum-existence property give rise to *conuclei* σ defined as above. Namely $\sigma : A \to A$ is an interior operator that further satisfies $\sigma(x) \cdot \sigma(y) \leq \sigma(xy)$ and $\sigma(1) = 1$. Then the structure $\mathbf{A}_\sigma = (\sigma[A], \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, 1)$ is a residuated lattice, where $\sigma[A] = D$, $x \wedge_\sigma y = \sigma(x \wedge y)$, $x \backslash_\sigma y = \sigma(x \backslash y)$ and $x /_\sigma y = \sigma(x/y)$. Moreover, B is a partial subalgebra of $\mathbf{A}_\sigma$. (For example, if $a, b, a \wedge b \in B$, then $a \wedge_\sigma b = a \wedge b$.)

The Tarski-McKinzei conucleus

So, if a variety of residuated lattices has locally finite monoid reducts then it has the FEP.

The Tarski-McKinzei conucleus

So, if a variety of residuated lattices has locally finite monoid reducts then it has the FEP. Examples are Heyting algebras (Tarski-McKinzei)

The Tarski-McKinzei conucleus

So, if a variety of residuated lattices has locally finite monoid reducts then it has the FEP. Examples are Heyting algebras (Tarski-McKinzei) and more generally commutative k-potent residuated lattices (Block-van Alten), $x^{k+1} = x^k$.

The Tarski-McKinzei conucleus

So, if a variety of residuated lattices has locally finite monoid reducts then it has the FEP. Examples are Heyting algebras (Tarski-McKinzei) and more generally commutative k-potent residuated lattices (Block-van Alten), $x^{k+1} = x^k$.

Note that k-potency is not a necessary condition for local finiteness of the monoid reduct, but *periodicity* ($x^m = x^n$ for some m, n) is; also commutative periodic (for some fixed m, n) residuated lattices have locally finite monoid reducts.

The Tarski-McKinzei conucleus

So, if a variety of residuated lattices has locally finite monoid reducts then it has the FEP. Examples are Heyting algebras (Tarski-McKinzei) and more generally commutative k-potent residuated lattices (Block-van Alten), $x^{k+1} = x^k$.

Note that k-potency is not a necessary condition for local finiteness of the monoid reduct, but *periodicity* ($x^m = x^n$ for some m, n) is; also commutative periodic (for some fixed m, n) residuated lattices have locally finite monoid reducts.

Commutativity is not a necessary condition, either. For example, together with periodicity, any equation (a) is sufficient instead of commutativity. The open question is which monoid equations together with periodicity yield local finiteness of the monoid reducts.

Theorem The variety of monoids satisfying a non-balanced (a) is locally finite. Consequently, any variety of residuated latices axiomatized by any non-balanced (a) (and possibly by any other $\{\vee, \cdot, 1\}$-identities) has the FEP.

Non-balanced (a)

Theorem The variety of monoids satisfying a non-balanced (a) is locally finite. Consequently, any variety of residuated latices axiomatized by any non-balanced (a) (and possibly by any other $\{\vee, \cdot, 1\}$-identities) has the FEP.

Proof sketch: By a known theorem it is enough to check the local finiteness of the varieties of groups and of nilsemigroups.

Non-balanced (a)

Theorem The variety of monoids satisfying a non-balanced (a) is locally finite. Consequently, any variety of residuated latices axiomatized by any non-balanced (a) (and possibly by any other $\{\vee, \cdot, 1\}$-identities) has the FEP.

Proof sketch: By a known theorem it is enough to check the local finiteness of the varieties of groups and of nilsemigroups.

For groups setting $y_k = 1$ for all k, we obtain $x^{r+1} = x^{\sum a_i}$; by cancellation we get $x^d = 1$, where $d = |r + 1 - \sum a_i|$.

Theorem The variety of monoids satisfying a non-balanced (a) is locally finite. Consequently, any variety of residuated latices axiomatized by any non-balanced (a) (and possibly by any other $\{\vee, \cdot, 1\}$-identities) has the FEP.

Proof sketch: By a known theorem it is enough to check the local finiteness of the varieties of groups and of nilsemigroups.

For groups setting $y_k = 1$ for all k, we obtain $x^{r+1} = x^{\sum a_i}$; by cancellation we get $x^d = 1$, where $d = |r + 1 - \sum a_i|$.

We can also obtain $x^{b_1} y_i x y_{i+1} x^{b_2} = x^{c_1} y_i y_{i+1} x^{c_2}$, which yields $y_i x y_{i+1} = x^{d_1} y_i y_{i+1} x^{d_2}$, by multiplying by the appropriate inverses from the appropriate side. (We omit the argument when the zero exponent is in one of the ends of the word.)

Non-balanced (a)

Theorem The variety of monoids satisfying a non-balanced (a) is locally finite. Consequently, any variety of residuated latices axiomatized by any non-balanced (a) (and possibly by any other $\{\vee, \cdot, 1\}$-identities) has the FEP.

Proof sketch: By a known theorem it is enough to check the local finiteness of the varieties of groups and of nilsemigroups.

For groups setting $y_k = 1$ for all k, we obtain $x^{r+1} = x^{\sum a_i}$; by cancellation we get $x^d = 1$, where $d = |r + 1 - \sum a_i|$.

We can also obtain $x^{b_1} y_i x y_{i+1} x^{b_2} = x^{c_1} y_i y_{i+1} x^{c_2}$, which yields $y_i x y_{i+1} = x^{d_1} y_i y_{i+1} x^{d_2}$, by multiplying by the appropriate inverses from the appropriate side. (We omit the argument when the zero exponent is in one of the ends of the word.)

Then we use this identity to obtain a normal form for each word on a finite number of generators $\{g_1, g_2, \ldots, g_k\}$, by successively moving each occurrence of the generators (one by one) to the ends of the word, possibly by changing the exponents, to obtain the form $g_1^{e_1} g_2^{e_2} \cdots g_k^{e_k} g_k^{f_k} \cdots g_2^{f_2} g_1^{f_1}$. Using the identity $x^d = 1$ we can actually assume that each of the e_i, f_i is at most d.

The family of Zimin words Z_n, for positive integer n, relative to a countably infinite list of variables $x_1, x_2, ..,$ is defined inductively by $Z_1 = x_1$ and $Z_{n+1} = Z_n x_{n+1} Z_n$.

Zimin-like equations

The family of Zimin words Z_n, for positive integer n, relative to a countably infinite list of variables $x_1, x_2, .,$ is defined inductively by $Z_1 = x_1$ and $Z_{n+1} = Z_n x_{n+1} Z_n$.We usually write x for x_1, y_1 for x_2 and in general y_i for x_{i+1}. So, for example, $Z_4 = xy_1xy_2xy_1xy_3xy_1xy_2xy_1x$.

The family of Zimin words Z_n, for positive integer n, relative to a countably infinite list of variables $x_1, x_2, .,$ is defined inductively by $Z_1 = x_1$ and $Z_{n+1} = Z_n x_{n+1} Z_n$.We usually write x for x_1, y_1 for x_2 and in general y_i for x_{i+1}. So, for example, $Z_4 = xy_1xy_2xy_1xy_3xy_1xy_2xy_1x$.

We consider equations such as

$$xy_1xy_2xy_1xy_3xy_1x = x^3y_1y_2x^2y_1x^5y_3xy_1x^3,$$

which are determined by the vector $\bar{a} = (3, 0, 2, 5, 1, 3)$ of the exponents of x on the right-hand side; we write $[\bar{a}]$ for this equation.

The family of Zimin words Z_n, for positive integer n, relative to a countably infinite list of variables $x_1, x_2, .,$ is defined inductively by $Z_1 = x_1$ and $Z_{n+1} = Z_n x_{n+1} Z_n$.We usually write x for x_1, y_1 for x_2 and in general y_i for x_{i+1}. So, for example, $Z_4 = xy_1xy_2xy_1xy_3xy_1xy_2xy_1x$.

We consider equations such as

$$xy_1xy_2xy_1xy_3xy_1x = x^3y_1y_2x^2y_1x^5y_3xy_1x^3,$$

which are determined by the vector $\bar{a} = (3, 0, 2, 5, 1, 3)$ of the exponents of x on the right-hand side; we write $[\bar{a}]$ for this equation.

Note that $[\bar{a}]$ is a substitution instance of $(\bar{a})$, so it is a weaker equation. If each variable (namely x) appears the same number of times in each side then we call it *balanced*.

The family of Zimin words Z_n, for positive integer n, relative to a countably infinite list of variables $x_1, x_2, .$, is defined inductively by $Z_1 = x_1$ and $Z_{n+1} = Z_n x_{n+1} Z_n$.We usually write x for x_1, y_1 for x_2 and in general y_i for x_{i+1}. So, for example, $Z_4 = xy_1xy_2xy_1xy_3xy_1xy_2xy_1x$.

We consider equations such as

$$xy_1xy_2xy_1xy_3xy_1x = x^3y_1y_2x^2y_1x^5y_3xy_1x^3,$$

which are determined by the vector $\bar{a} = (3, 0, 2, 5, 1, 3)$ of the exponents of x on the right-hand side; we write $[\bar{a}]$ for this equation.

Note that $[\bar{a}]$ is a substitution instance of $(\bar{a})$, so it is a weaker equation. If each variable (namely x) appears the same number of times in each side then we call it *balanced*. Note that if the equation is not balanced then it implies periodicity.

The family of Zimin words Z_n, for positive integer n, relative to a countably infinite list of variables $x_1, x_2, .,$ is defined inductively by $Z_1 = x_1$ and $Z_{n+1} = Z_n x_{n+1} Z_n$.We usually write x for x_1, y_1 for x_2 and in general y_i for x_{i+1}. So, for example, $Z_4 = xy_1xy_2xy_1xy_3xy_1xy_2xy_1x$.

We consider equations such as

$$xy_1xy_2xy_1xy_3xy_1x = x^3y_1y_2x^2y_1x^5y_3xy_1x^3,$$

which are determined by the vector $\bar{a} = (3, 0, 2, 5, 1, 3)$ of the exponents of x on the right-hand side; we write $[\bar{a}]$ for this equation.

Note that $[\bar{a}]$ is a substitution instance of $(\bar{a})$, so it is a weaker equation. If each variable (namely x) appears the same number of times in each side then we call it *balanced*. Note that if the equation is not balanced then it implies periodicity.

Theorem. If a monoid variety satisfies a balanced equation $[a]$ and a periodic identity then it is locally finite.

Zimin-like equations

Theorem. If a monoid variety satisfies a balanced equation $[a]$ and a periodic identity then it is locally finite.

Theorem. If a monoid variety satisfies a balanced equation $[a]$ and a periodic identity then it is locally finite.

Proof. Note that the right-hand side of the equation $[a]$ contains the square of x.

Theorem. If a monoid variety satisfies a balanced equation $[a]$ and a periodic identity then it is locally finite.

Proof. Note that the right-hand side of the equation $[a]$ contains the square of x.
By the way of contradiction, assume that a periodic monoid M generated by a finite set of generators X is infinite and satisfies $Z_\ell = W$.

Zimin-like equations

Theorem. If a monoid variety satisfies a balanced equation $[a]$ and a periodic identity then it is locally finite.

Proof. Note that the right-hand side of the equation $[a]$ contains the square of x.
By the way of contradiction, assume that a periodic monoid M generated by a finite set of generators X is infinite and satisfies $Z_\ell = W$. By results in symbolic dynamics/combinatorics on words there exists a *uniformly recurrent* bi-infinite word β over X, where all its finite subwords are *geodesics*.

Theorem. If a monoid variety satisfies a balanced equation $[a]$ and a periodic identity then it is locally finite.

Proof. Note that the right-hand side of the equation $[a]$ contains the square of x.
By the way of contradiction, assume that a periodic monoid M generated by a finite set of generators X is infinite and satisfies $Z_\ell = W$. By results in symbolic dynamics/combinatorics on words there exists a *uniformly recurrent* bi-infinite word β over X, where all its finite subwords are *geodesics*. By another result in symbolic dynamics, for every letter a in X, there must be a finite subword u of β such that $Z_\ell = W$ implies $u = sa^m t$ for some words s and t, where $x^m = x^n, m > n$ is the periodic identity.

Theorem. If a monoid variety satisfies a balanced equation $[a]$ and a periodic identity then it is locally finite.

Proof. Note that the right-hand side of the equation $[a]$ contains the square of x.
By the way of contradiction, assume that a periodic monoid M generated by a finite set of generators X is infinite and satisfies $Z_\ell = W$. By results in symbolic dynamics/combinatorics on words there exists a *uniformly recurrent* bi-infinite word β over X, where all its finite subwords are *geodesics*. By another result in symbolic dynamics, for every letter a in X, there must be a finite subword u of β such that $Z_\ell = W$ implies $u = sa^m t$ for some words s and t, where $x^m = x^n, m > n$ is the periodic identity. Given that $Z_\ell = W$ is balanced, then $u = sa^m t$ is balanced, i.e. u and $sa^m t$ have the same length.

Theorem. If a monoid variety satisfies a balanced equation $[a]$ and a periodic identity then it is locally finite.

Proof. Note that the right-hand side of the equation $[a]$ contains the square of x.
By the way of contradiction, assume that a periodic monoid M generated by a finite set of generators X is infinite and satisfies $Z_\ell = W$. By results in symbolic dynamics/combinatorics on words there exists a *uniformly recurrent* bi-infinite word β over X, where all its finite subwords are *geodesics*. By another result in symbolic dynamics, for every letter a in X, there must be a finite subword u of β such that $Z_\ell = W$ implies $u = sa^m t$ for some words s and t, where $x^m = x^n, m > n$ is the periodic identity. Given that $Z_\ell = W$ is balanced, then $u = sa^m t$ is balanced, i.e. u and $sa^m t$ have the same length. By periodicity, $sa^m t = sa^n t$, which means that $u = sa^m t$ is not a geodesic, contradicting the fact that u is a geodesic.

Non-balanced [a]

We have suficient conditions for local finiteness of the monoid reduct.

Non-balanced [a]

We have suficient conditions for local finiteness of the monoid reduct.

Notation: We denote by t the highest-index (middle) variable and by $x^a t x^a = x^b t x^c$ the equation obtained by setting $y_i = 1$ except for x and t. Also we set $d = b + c - 2a$ and $g = gcd\{d, |b - a|\}$.

Non-balanced [a]

We have suficient conditions for local finiteness of the monoid reduct.

Notation: We denote by t the highest-index (middle) variable and by $x^a t x^a = x^b t x^c$ the equation obtained by setting $y_i = 1$ except for x and t. Also we set $d = b + c - 2a$ and $g = gcd\{d, |b - a|\}$.

Theorem. If $g \neq d$, and x^g appears at least d/g many times in w, then we have local finiteness. In particular, if $g = 1$ or 2, we obtain local finiteness.

Non-balanced [a]

We have suficient conditions for local finiteness of the monoid reduct.

Notation: We denote by t the highest-index (middle) variable and by $x^a t x^a = x^b t x^c$ the equation obtained by setting $y_i = 1$ except for x and t. Also we set $d = b + c - 2a$ and $g = gcd\{d, |b-a|\}$.

Theorem. If $g \neq d$, and x^g appears at least d/g many times in w, then we have local finiteness. In particular, if $g = 1$ or 2, we obtain local finiteness.

(Among other things, we make use the fact that in cancellative monoids if $x^d t = t x^d$ and $x^e t = t x^e$, then $x^f t = t x^f$, where $f = \gcd\{d, e\}$.)

Non-balanced [a]

We have suficient conditions for local finiteness of the monoid reduct.

Notation: We denote by t the highest-index (middle) variable and by $x^a t x^a = x^b t x^c$ the equation obtained by setting $y_i = 1$ except for x and t. Also we set $d = b + c - 2a$ and $g = gcd\{d, |b - a|\}$.

Theorem. If $g \neq d$, and x^g appears at least d/g many times in w, then we have local finiteness. In particular, if $g = 1$ or 2, we obtain local finiteness.

(Among other things, we make use the fact that in cancellative monoids if $x^d t = t x^d$ and $x^e t = t x^e$, then $x^f t = t x^f$, where $f = \gcd\{d, e\}$.)

For example in our equation $xyxzxyx = yxzx^4yx$ we have $d = 1 + 5 - 2 \cdot 2 = 2$ and $g = gcd\{2, 5 - 2\} = 1$, so we have the FEP.

Connections to the Burnside problem and other cases

The equation $xyx = x^{a+1}yx$is equivalent to $x^a = 1$ in the theory of groups. The variety of monoids axiomatized by $xyx = x^{a+1}yx$ will be locally finite if and only if the corresponding variety of groups is locally finite.

Connections to the Burnside problem and other cases

The equation $xyx = x^{a+1}yx$is equivalent to $x^a = 1$ in the theory of groups. The variety of monoids axiomatized by $xyx = x^{a+1}yx$ will be locally finite if and only if the corresponding variety of groups is locally finite.

The variety of groups axiomatixed by the equation $x^n = 1$ is locally finite for the values $n \in \{1, 2, 3, 4, 6\}$.

Connections to the Burnside problem and other cases

The equation $xyx = x^{a+1}yx$is equivalent to $x^a = 1$ in the theory of groups. The variety of monoids axiomatized by $xyx = x^{a+1}yx$ will be locally finite if and only if the corresponding variety of groups is locally finite.

The variety of groups axiomatixed by the equation $x^n = 1$ is locally finite for the values $n \in \{1, 2, 3, 4, 6\}$. It is not locally finite for $n \geq 665$ and odd, and also for $n \geq 2^{48}$.

Connections to the Burnside problem and other cases

The equation $xyx = x^{a+1}yx$is equivalent to $x^a = 1$ in the theory of groups. The variety of monoids axiomatized by $xyx = x^{a+1}yx$ will be locally finite if and only if the corresponding variety of groups is locally finite.

The variety of groups axiomatixed by the equation $x^n = 1$ is locally finite for the values $n \in \{1, 2, 3, 4, 6\}$. It is not locally finite for $n \geq 665$ and odd, and also for $n \geq 2^{48}$.

Fact. The variety of monoids axiomatized by $xyx = x^{a+1}yx^{b+1}$ is locally finite if $\gcd(a, b) = 1$.

Connections to the Burnside problem and other cases

The equation $xyx = x^{a+1}yx$is equivalent to $x^a = 1$ in the theory of groups. The variety of monoids axiomatized by $xyx = x^{a+1}yx$ will be locally finite if and only if the corresponding variety of groups is locally finite.

The variety of groups axiomatixed by the equation $x^n = 1$ is locally finite for the values $n \in \{1, 2, 3, 4, 6\}$. It is not locally finite for $n \geq 665$ and odd, and also for $n \geq 2^{48}$.

Fact. The variety of monoids axiomatized by $xyx = x^{a+1}yx^{b+1}$ is locally finite if $\gcd(a, b) = 1$.

Fact. The variety of monoids axiomatized by

$$Z_3 = x^{a_0}yx^{a_1}zx^{a_2}yx^{a_3},$$

is locally finite when $a_0a_1a_2a_3 = 0$.

Undecidability

Based on a square-free language L, R. Horčik constructs a residuated frame $\mathbf{W}$ and shows that every variety of residuated lattices that contains the associated Galois residuated lattice $\mathbf{W}^+$ has undecidable word problem. It is observed that $\mathbf{W}^+$ satisfies the identities $x \leq x^2 = x^3$.

Undecidability

Based on a square-free language L, R. Horčik constructs a residuated frame $\mathbf{W}$ and shows that every variety of residuated lattices that contains the associated Galois residuated lattice $\mathbf{W}^+$ has undecidable word problem. It is observed that $\mathbf{W}^+$ satisfies the identities $x \leq x^2 = x^3$. We can further prove that $\mathbf{W}^+$ satisfies:

1. $x^2 = x$ if $x \in \{\bot, 1, \top\}$, otherwise $x^2 = \top$.
2. $x^2 y = y x^2$.
3. $x\top = \top x = \top$, for $x \neq \bot$.
4. $(xy)^2 = x^2 y^2 = (yx)^2$.

Undecidability

Based on a square-free language L, R. Horčik constructs a residuated frame $\mathbf{W}$ and shows that every variety of residuated lattices that contains the associated Galois residuated lattice $\mathbf{W}^+$ has undecidable word problem. It is observed that $\mathbf{W}^+$ satisfies the identities $x \leq x^2 = x^3$. We can further prove that $\mathbf{W}^+$ satisfies:

1. $x^2 = x$ if $x \in \{\bot, 1, \top\}$, otherwise $x^2 = \top$.
2. $x^2 y = y x^2$.
3. $x\top = \top x = \top$, for $x \neq \bot$.
4. $(xy)^2 = x^2 y^2 = (yx)^2$.

If x appears in v and $w = u_1 v^2 u_2$, then x^2 appears in w (distribution of squares), and it can be moved successively (by commutation of squares) to be adjacent to every other occurrence of x in w, and result to a power of x, which can then be reduced to x^2 (by $x^3 = x^2$), resulting in all occurences of x in w to be collected together in the beginning of w in the form of a single x^2. We do this for all variables of w appearing under a square in it and we call this initial part of w as w_s; the remaining part of w is called w_f. Thus $w = w_s w_f$ follows by the above equations.

Undecidability

Based on a square-free language L, R. Horčik constructs a residuated frame $\mathbf{W}$ and shows that every variety of residuated lattices that contains the associated Galois residuated lattice $\mathbf{W}^+$ has undecidable word problem. It is observed that $\mathbf{W}^+$ satisfies the identities $x \leq x^2 = x^3$. We can further prove that $\mathbf{W}^+$ satisfies:

1. $x^2 = x$ if $x \in \{\bot, 1, \top\}$, otherwise $x^2 = \top$.
2. $x^2 y = y x^2$.
3. $x\top = \top x = \top$, for $x \neq \bot$.
4. $(xy)^2 = x^2 y^2 = (yx)^2$.

If x appears in v and $w = u_1 v^2 u_2$, then x^2 appears in w (distribution of squares), and it can be moved successively (by commutation of squares) to be adjacent to every other occurrence of x in w, and result to a power of x, which can then be reduced to x^2 (by $x^3 = x^2$), resulting in all occurences of x in w to be collected together in the beginning of w in the form of a single x^2. We do this for all variables of w appearing under a square in it and we call this initial part of w as w_s; the remaining part of w is called w_f. Thus $w = w_s w_f$ follows by the above equations. So $xxyxzxyx = xyxzx^4yx$ becomes $x^2yzy = x^2yzy$ which holds in $\mathbf{W}^+$.

Undecidability

Most of the equations discussed here were purely monoid equations.

Undecidability

Most of the equations discussed here were purely monoid equations. Knotted equations $x^m \leq x^n$ involve join $x^m \vee x^n = x^n$.

Undecidability

Most of the equations discussed here were purely monoid equations. Knotted equations $x^m \leq x^n$ involve join $x^m \vee x^n = x^n$. Every equation over $\{\vee, \cdot, 1\}$ can be written as a conjunction of *simple* equations of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are monoidal terms.

Most of the equations discussed here were purely monoid equations. Knotted equations $x^m \leq x^n$ involve join $x^m \vee x^n = x^n$. Every equation over $\{\vee, \cdot, 1\}$ can be written as a conjunction of *simple* equations of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are monoidal terms. Such an example is $x \leq x^2 \vee x^3$.

Undecidability

Most of the equations discussed here were purely monoid equations. Knotted equations $x^m \leq x^n$ involve join $x^m \vee x^n = x^n$. Every equation over $\{\vee, \cdot, 1\}$ can be written as a conjunction of *simple* equations of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are monoidal terms. Such an example is $x \leq x^2 \vee x^3$.

In joint work with G. StJohn we can prove that the word problem of the variety axiomatized by one of these equations, even with the addition of commutativity, is not primitive-recursively decidable. (We suspect it is undecidable.)

Undecidability

Most of the equations discussed here were purely monoid equations. Knotted equations $x^m \leq x^n$ involve join $x^m \vee x^n = x^n$. Every equation over $\{\vee, \cdot, 1\}$ can be written as a conjunction of *simple* equations of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_i are monoidal terms. Such an example is $x \leq x^2 \vee x^3$.

In joint work with G. StJohn we can prove that the word problem of the variety axiomatized by one of these equations, even with the addition of commutativity, is not primitive-recursively decidable. (We suspect it is undecidable.) For the latter even the equational theory is not primitive-recursively decidable.