A Modal Logic of the Real Numbers

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Many-valued modal logics extend the Kripke frame setting of classical modal logic with a many-valued semantics at each world to model modal notions such as necessity, belief, and spatio-temporal relations in the presence of uncertainty, possibility, or vagueness. Although it can be quite straightforward to define a suitable many-valued modal logic for a particular application, obtaining a deeper mathematical understanding of such a logic via an adequate axiomatization or algebraic semantics is often not so easy. In particular, only an axiomatization with an infinitary rule has been provided to date for the basic modal infinite-valued Łukasiewicz logic [4] (see also [3, 1]). As an intermediary step towards finding a finitary axiomatization and thereby also an algebraic semantics for this logic, we consider here a simple prototype many-valued modal logic of "magnitude" (as opposed to the "order-based" many-valued modal logics studied in [2]) that extends the multiplicative fragment of abelian logic (see, e.g., [5]).

Let us denote by Fm the set of formulas defined inductively over a countably infinite set Var of propositional variables using the binary connective $\to$ and modal connective $\Box$. We define

$$\overline{0} := p_0 \to p_0, \quad \neg\varphi := \varphi \to \overline{0}, \quad \varphi \& \psi := \neg\varphi \to \psi, \quad \text{and} \quad \Diamond\varphi := \neg\Box\neg\varphi,$$

and let $0\varphi = \overline{0}$ and $(n+1)\varphi = \varphi\&(n\varphi)$ for each $n \in \mathbb{N}$.

A *frame* is a pair $\mathfrak{F} = \langle W, R\rangle$, where W is a non-empty set of *worlds* and $R \subseteq W \times W$ is an *accessibility relation*. $\mathfrak{F}$ is called *serial* if for all $x \in W$, there exists $y \in W$ such that Rxy. A K($\mathbb{R}$)-*model* $\mathfrak{M} = \langle W, R, V\rangle$ consists of a serial frame $\langle W, R\rangle$ and a map $V\colon \mathrm{Var} \times W \to [-r, r]$ for some $r \in \mathbb{R}^+$, called a *valuation*. This valuation is extended to $V\colon \mathrm{Fm} \times W \to \mathbb{R}$ by

$$\begin{aligned} V(\varphi \to \psi, x) &= V(\psi, x) - V(\varphi, x) \\ V(\Box\varphi, x) &= \bigwedge\{V(\varphi, y) : Rxy\}. \end{aligned}$$

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It follows also that

$$\begin{array}{rclcrcl} V(\overline{0}, x) & = & 0 & \qquad & V(\varphi \& \psi, x) & = & V(\varphi, x) + V(\psi, x) \\ V(\neg\varphi, x) & = & -V(\varphi, x) & \qquad & V(\Diamond\varphi, x) & = & \bigvee\{V(\varphi, y) : Rxy\}. \end{array}$$

A formula $\varphi \in \mathrm{Fm}$ will be called *valid* in a K($\mathbb{R}$)-model $\mathfrak{M} = \langle W, R, V\rangle$ if $V(\varphi, x) \geq 0$ for all $x \in W$. If φ is valid in all K($\mathbb{R}$)-models, then φ is said to be K($\mathbb{R}$)-*valid*, written $\models_{\mathrm{K}(\mathbb{R})} \varphi$.

In Fig. 1 we propose an axiom system K($\mathbb{R}$) for this logic that extends the multiplicative fragment of abelian logic axiomatized by (B), (C), (I), (A), and (mp). It is easily shown that any formula derivable in this system is K($\mathbb{R}$)-valid. For the converse direction, we make use of the analytic sequent calculus presented in Fig. 2, where a sequent $\Gamma \Rightarrow \Delta$ is an ordered pair of finite multisets of formulas, $k\Gamma$ denotes $\Gamma, \ldots, \Gamma$ (k times), and $\Box\Gamma$ denotes the multiset of boxed formulas $[\Box\varphi : \varphi \in \Gamma]$. Consider the interpretation (where $\varphi_1 \& \ldots \& \varphi_n = \overline{0}$ for $n = 0$):

$$\mathcal{I}(\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m) := (\varphi_1 \& \ldots \& \varphi_n) \to (\psi_1 \& \ldots \& \psi_m).$$

We prove first that $\Gamma \Rightarrow \Delta$ is derivable in GK($\mathbb{R}$) if and only if $\mathcal{I}(\Gamma \Rightarrow \Delta)$ is derivable in K($\mathbb{R}$) and that GK($\mathbb{R}$) admits cut elimination. Completeness of the calculus GK($\mathbb{R}$) (a more challenging step) is then established using an intermediate labelled calculus to produce systems of linear inequations over $\mathbb{R}$. We hence obtain the main result:

Theorem 1. *The following are equivalent for any formula φ:*

(1) φ *is* K($\mathbb{R}$)-*valid.*

(2) φ *is derivable in* K($\mathbb{R}$)

(3) $\Rightarrow \varphi$ *is derivable in* GK($\mathbb{R}$).

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(B)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	(A)	$((\varphi \to \psi) \to \psi) \to \varphi$
(C)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
(I)	$\varphi \to \varphi$	(D_n)	$\Box(n\varphi) \to n\Box\varphi \qquad (n \geq 2)$

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\text{mp}) \qquad \frac{\varphi}{\Box\varphi}\ (\text{nec}) \qquad \frac{n\varphi}{\varphi}\ (\text{con}_n) \quad (n \geq 2)$$

Figure 1: The axiom system $K(\mathbb{R})$

$$\frac{}{\Delta \Rightarrow \Delta}\ (\textsc{id}) \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Pi \Rightarrow \varphi, \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\textsc{cut})$$

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\textsc{mix}) \qquad \frac{n\Gamma \Rightarrow n\Delta}{\Gamma \Rightarrow \Delta}\ (\textsc{sc}_n) \quad (n \geq 2)$$

$$\frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)$$

$$\frac{\Gamma \Rightarrow n[\varphi]}{\Box\Gamma \Rightarrow n[\Box\varphi]}\ (\Box_n) \quad (n \geq 0)$$

Figure 2: The sequent calculus $GK(\mathbb{R})$