Riesz MV-algebras and Divisible MV-algebras: logic, analysis and polyhedral geometry

Antonio Di Nola[1], Serafina Lapenta[1], and Ioana Leuştean[2]

[1] Department of Mathematics, University of Salerno Fisciano, Salerno, Italy
adinola@unisa.it, slapenta@unisa.it

[2] Department of Computer Science, Faculty of Science University of Bucharest, Bucharest, Romania
ioana@fmi.unibuc.ro

Riesz MV-algebras [7] are defined by enriching the MV-algebras with a scalar multiplication with scalars from $[0, 1]$. When the scalars are rational numbers, one gets a class of structures that are term-equivalent to Divisible MV-algebras (DMV-algebras, for short) [9]. We recall [6, 11] for all unexplained notions.

We will present the connections between Łukasiewicz logic, the logic $\mathcal{Q}$Ł of Divisible MV-algebras and the logic $\mathcal{R}$Ł of Riesz MV-algebras.

We start by investigating $\mathcal{R}$Ł with methods from analysis. After the definition of *limit* of formulas, we prove the following theorem.

Theorem 1. *Any formula of $\mathcal{R}$Ł is the limit of a sequence of rational formulas.*

Then, we consider the notion of *integral* of a formula and connect such notion with the invariant Lebesgue state defined by Mundici and Panti [11, 12]. Moreover, we obtain Pavelka-style completeness for $\mathcal{R}$Ł, namely the following proposition.

Proposition 1 (Pavelka completeness)**.** *If φ is a formula of $\mathcal{R}$Ł, then*

$$\max\{r \in [0,1] \mid \vdash \delta_r \to \varphi\} = \mid \varphi \mid = \parallel \varphi \parallel = \min\{e(\varphi) \mid e \textit{ is an evaluation}\}.$$

As a second approach, we present a categorical equivalence between *categories of presentations* for MV-algebras and Divisible MV-algebras. By categories of presentations we mean categories whose objects are couples (X, I) where X is a set of generators and I is a principal ideal in the free MV-algebra generated by X or in the free Divisible MV-algebra generated by X. The maps are $\varphi : (X, I) \to (Y, J)$ such that $\varphi : Free(X) \to Free(Y)$, $\varphi(I) \subseteq J$. For arrows between presentations of DMV-algebras we shall add the condition $\varphi(X) \subseteq Free_{MV}(Y)(\subseteq Free_{DMV}(Y))$.

The final approach is devoted to the connection between MV-algebras and the theory of polyhedra. We refer to [11, 3, 4, 5, 10, 8] for recent advances on the subject. The key players are finitely presented MV-algebras, i.e. quotients of a finitely generated free MV-algebra by a principal ideal. Such algebras are in duality (from a categorical point of view) with rational polyhedra, i.e. polyhedra whose vertices have rational coordinates.

A recent work of Di Nola et al. [8] has shown that, retracing the steps of [10] (and hence the Baker and Beynon original duality for Riesz Spaces and polyhedra [1, 2]), we can obtain a similar result for finitely presented Riesz MV-algebras and polyhedra in some $[0,1]^n$. The following theorem uses the results in [8, 10] to prove a duality for finitely presented DMV-algebras and rational polyhedra. Moreover, we prove that an algebra is finitely generated and projective iff so it is the corresponding lattice-ordered structure (ℓ-group or Riesz Space) associated.

Theorem 2. *(i) The category of rational polyhedra in some* $[0,1]^n$ *is equivalent to the dual of the category of finitely presented Divisible MV-algebras.*

(ii) The category of rational polyhedra in some $\mathbb{R}^n$ *is equivalent to the dual of the category of finitely presented Divisible MV-algebras.*

Finally, we analyze the behaviour of finitely presented, finitely generated and projective MV-algebras and Riesz MV-algebras with respect to the tensor product $\otimes_{MV}$ of MV-algebras. We recall that when we deal with semisimple structures, the tensor product, together with the usual forgetful functor, provides an adjunction between semisimple MV-algebras and semisimple Riesz MV-algebras. Although such adjunction do not restrict to finitely presented, finitely generated nor projective algebras, we have the following.

Proposition 2. *Let* A *be a semisimple MV-algebra. If* A *is finitely generated (resp. finitely presented, projective), then* $[0,1] \otimes_{MV} A$ *is finitely generated (resp. finitely presented, projective).*

References

[1] Baker K.A., *Free vector lattices*, Canad. J. Math 20 (1968) 58-66.

[2] Beynon W. M., *Duality theorem for finitely generated vector lattices*, Proc. London Math. Soc. (3) 31 (1975) 114-128.

[3] Cabrer L.M., Mundici D., *Projective MV-algebras and rational polyhedra*, Algebras Universalis 62(1) 2009 63-74.

[4] Cabrer L.M., Mundici D., *Rational polyhedra and projective lattive ordered abelian groups with strong unit*, Communications in Contemporary Mathematics 14(3) DOI: 10.1142/S0219199712500174.

[5] Cabrer L.M., Mundici D., *Finitely presented lattice-ordered abelian groups with order unit*, Journal of Algebra, 343 (2011) 1-10

[6] Cignoli R., D'Ottaviano I.M.L., Mundici D., *Algebraic foundation of many valued Reasoning*, Kluver Academc Publ (2000) Dordrecht.

[7] Di Nola A., Leuştean I., *Łukasiewicz logic and Riesz Spaces*, Soft Computing 18 (12) (2014) 2349-2363.

[8] Di Nola A., Lenzi G., Vitale G., *Riesz MV-algebras of nonlinear functions*, UNILOG2015.

[9] Gerla B., *Rational Łukasiewicz logic and Divisible MV-algebras*, Neural Networks World 10 (2001).

[10] Marra V., Spada L., *Duality, projectivity and unification in Łukasiewicz logic and MV-algebras*, Annals of Pure and Applied logic 164(3) (2013) 192-210.

[11] Mundici D., *Advances in Łukasiewicz calculus and MV-algebras*, Trends in Logic 35 Springer (2011).

[12] Panti G., *Invariant measures in free MV-algebras*, Communications in Algebras 36 (2008) 2849-2861.