Bilattices with two chains of truth values

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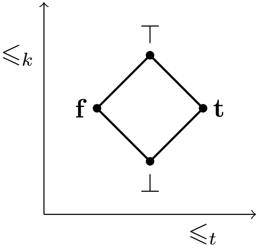
Outline

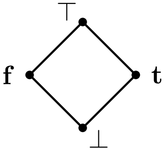
- Bilattices, interlaced bilattices and their product representation
- Default bilattices and the bilattices $\mathbf{J}_n$
- Natural duality theory
- Natural duality for the quasivarieties $\mathbb{ISP}(\mathbf{J}_n)$

Bilattices were introduced by Ginsberg in 1986 as a generalisation of Belnap's four valued logic from the mid 1970's.

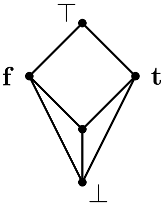


The vertical axis represents the *knowledge* order and the horizontal axis represents the *truth* order. The lattice operations $\otimes$ and $\oplus$ of the knowledge order represent *consensus* and *gullibility*.
In addition to two sets of lattice operations, a bilattice has a unary negation operation $\neg$ which *preserves* $\leqslant_k$ and *reverses* $\leqslant_t$.

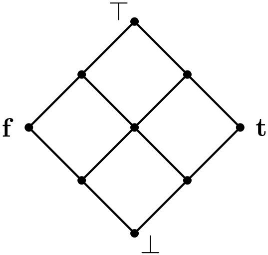
Examples of bilattices:



$\mathcal{FOUR}$



$\mathcal{FIVE}$



$\mathcal{NINE}$

A bilattice $\mathbf{B}$ has the algebraic signature

$$\mathbf{B} = \langle B; \otimes, \oplus, \wedge, \vee, \neg \rangle.$$

This signature will often contain bounds of one (or both) of the orders. The knowledge bounds are $\perp, \top$ and the truth bounds are $\mathbf{f}, \mathbf{t}$. Note: often one set of bounds will be term-definable from the other.

There can be varying levels of interaction between the orders. A bilattice $\mathbf{B}$ is *distributive* if for all $a, b, c \in B$ the identity

$$a \bullet (b * c) \approx (a \bullet b) * (a \bullet c)$$

holds for $\bullet, * \in \{\otimes, \oplus, \wedge, \vee\}$.

A bilattice is *interlaced* if

$$\begin{aligned} a \leqslant_t b &\implies a \otimes c \leqslant_t b \otimes c, \\ a \leqslant_t b &\implies a \oplus c \leqslant_t b \oplus c, \\ a \leqslant_k b &\implies a \wedge c \leqslant_k b \wedge c, \\ a \leqslant_k b &\implies a \vee c \leqslant_k b \vee c. \end{aligned}$$

NB: an interlaced bilattice $\mathbf{B}$ has $\leqslant_k$ as a subalgebra of $\mathbf{B}^2$.

Product bilattices

Let $\mathbf{L} = \langle L; \sqcap, \sqcup \rangle$ be a lattice. The operations of the product bilattice

$$\mathbf{L} \odot \mathbf{L} = \langle L \times L; \otimes, \oplus, \wedge, \vee, \neg \rangle$$

are defined for $(a, b), (c, d) \in L \times L$ by

$$\begin{aligned}
(a, b) \otimes (c, d) &= (a \sqcap c, b \sqcap d) \\
(a, b) \oplus (c, d) &= (a \sqcup c, b \sqcup d) \\
(a, b) \wedge (c, d) &= (a \sqcap c, b \sqcup d) \\
(a, b) \vee (c, d) &= (a \sqcup c, b \sqcap d) \\
\neg(a, b) &= (b, a).
\end{aligned}$$

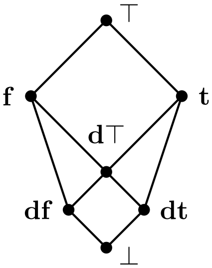
Think of an element $(a, b) \in L \times L$ as encoding evidence about some sentence: a is the evidence *for*, and b is the evidence *against*.

The following theorem was proven by various researchers at different levels of generality: Romanowska and Trakul (1989), Fitting (1990), Avron (1996), Rivieccio (2010) and others.

Theorem

A (bounded) bilattice $\mathbf{B}$ *is interlaced if and only if it is isomorphic to the bilattice product* $\mathbf{L} \odot \mathbf{L}$ *for some (bounded) lattice* $\mathbf{L}$.

Ginsberg proposed the bilattice $\mathcal{SEVEN}$ for inference in default logic. The additional truth values $\mathbf{dt}$ and $\mathbf{df}$ represent *"true by default"* and *"false by default"*.



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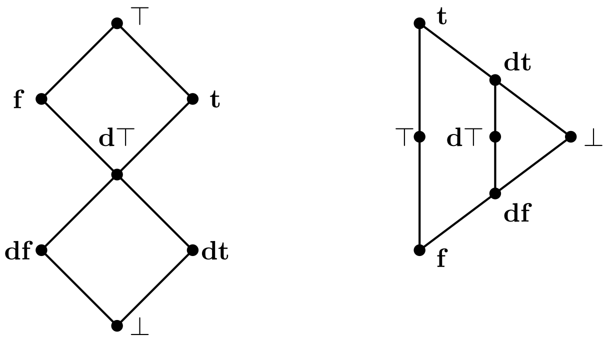


$\leqslant_k$ $\mathcal{SEVEN}$ $\leqslant_t$

Note: $\mathcal{SEVEN}$ is not interlaced, as $\mathbf{d}\top \leqslant_k \mathbf{t}$ but $\mathbf{d}\top \wedge \bot = \mathbf{df} \not\leqslant_k \bot = \mathbf{t} \wedge \bot$.

A hierarchy of default bilattices Cabrer, Craig, Priestley (2015)

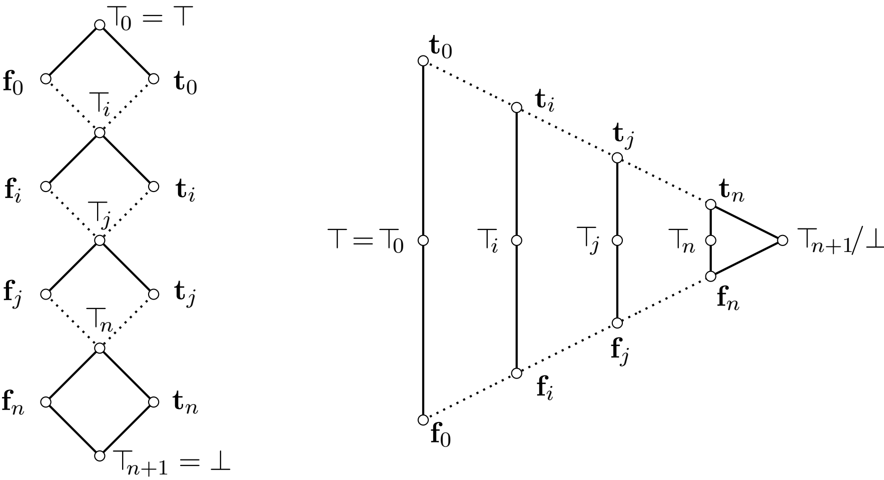


Figure: K_n in its knowledge order (left) and truth order (right); here $0 < i < j < n$

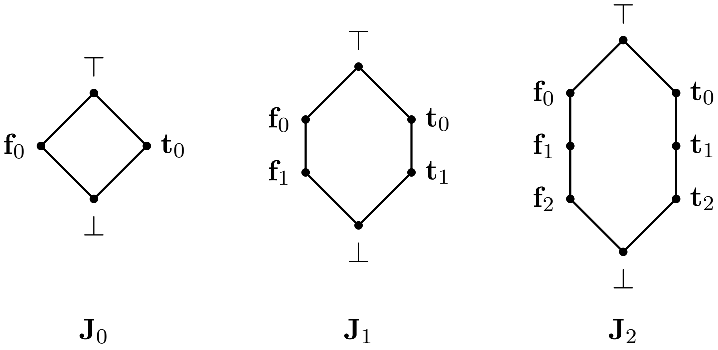


Figure: The bilattices $\mathbf{J}_0, \mathbf{J}_1$ and $\mathbf{J}_2$, drawn with their knowledge order.

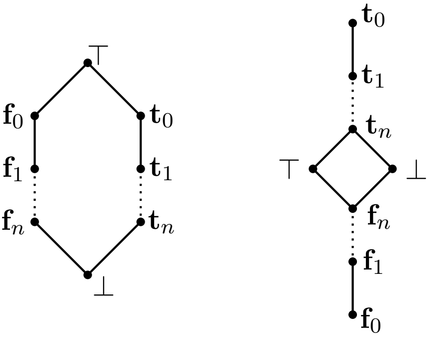


Figure: The bilattice $\mathbf{J}_n$ drawn in both its knowledge (left) and truth (right) orders.

Note: $\mathbf{J}_n$ is not interlaced for $n \geqslant 1$. We have $\mathbf{f}_0 \leqslant_k \top$ but

$$\bot \wedge \mathbf{f}_0 = \mathbf{f}_0 \nleqslant_k \mathbf{f}_n = \top \wedge \bot$$

Natural duality

Natural duality theory provides a uniform method for obtaining dual structures for algebras in a finitely generated quasivariety.

The theory has been developed by Clark, Davey, Werner, Priestley, Haviar and others.

It has been successfully applied to many (quasi)varieties related to logic such as: distributive lattices (with additional operations), de Morgan algebras, Kleene algebras, finitely-generated quasivarieties of Heyting algebras.

Let $\mathbf{M}$ be a finite algebra and consider the quasivariety $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$. For an algebra $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, we consider the dual, $\mathrm{D}(\mathbf{A})$, of $\mathbf{A}$ to be the set of homomorphisms $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$.

Examples:

Let $\mathbf{M} = \langle \{0, 1\}; \wedge, \vee, \neg, 0, 1\rangle$ (the two-element Boolean algebra) and $\boldsymbol{\mathcal{B}} = \mathbb{ISP}(\mathbf{M})$. Then for any Boolean algebra $\mathbf{A}$, we have that the set $\boldsymbol{\mathcal{B}}(\mathbf{A}, \mathbf{M})$ is in a one-to-one correspondence with the set of ultrafilters of $\mathbf{A}$.

For $\mathbf{M} = \langle \{0, 1\}; \wedge, \vee, 0, 1\rangle$ and $\boldsymbol{\mathcal{D}} = \mathbb{ISP}(\mathbf{M})$ (the variety of bounded distributive lattices) we have that for any $\mathbf{A} \in \boldsymbol{\mathcal{D}}$, the set $\boldsymbol{\mathcal{D}}(\mathbf{A}, \mathbf{M})$ is in bijective correspondence with the set of prime filters of $\mathbf{A}$. This is the usual underlying set for the dual space of $\mathbf{A}$ when using Priestley duality.

Let $\mathbf{M}$ be a finite algebra and consider the quasivariety $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$. Now let $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. There is a natural map

$$e \colon A \to M^{\boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M})} \text{ given by } (e(a))(f) = f(a)$$

for all $a \in A$ and all $f \in \boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M})$. The maps $e(a) \colon \boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M}) \to M$ are maps given by evaluation.

However, there are many other maps in $M^{\boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M})}$. Natural duality theory adds topological and relational structure to the sets $\boldsymbol{\mathcal{A}}(\mathbf{A},\mathbf{M})$ and M so that the only structure-preserving maps are the evaluation maps.

This gives a concrete representation of the algebra as a set of structure-preserving maps.

The structure of dual spaces

Given $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathbf{M})$, we search for a suitable *alter ego* for $\mathbf{M}$. We want a structure $\underset{\sim}{\mathbf{M}}$ (the set M equipped with topology and relational structure) such that $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ is a category which is dually equivalent to $\boldsymbol{\mathcal{A}}$.

For finite k, we say that a k-ary relation R on $\mathbf{M}$ is an *algebraic relation* if R is a subalgebra of $\mathbf{M}^k$. In order for the theory to work, the relational structure on the alter ego must consist of *finitary algebraic relations*.

We denote the alter ego by $\underset{\sim}{\mathbf{M}}$. That is,

$$\underset{\sim}{\mathbf{M}} := \langle M; \mathcal{R}, \mathcal{T} \rangle$$

where $\mathcal{T}$ is the discrete topology, $\mathcal{R}$ is a set of algebraic relations on $\mathbf{M}$.

Observe that $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}) \subseteq M^A$. It inherits the subspace topology from the product space $\underset{\sim}{\mathbf{M}}^A$ and as it is a closed subspace, this topology is compact.

The set $\boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ also inherits relational structure from $\underset{\sim}{\mathbf{M}}$ in the following way for $R \in \mathcal{R}$. For $f, g \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$

$$(f, g) \in R_{\mathbf{A}} \iff \forall\, a \in A \quad (f(a), g(a)) \in R.$$

Finally, the algebra $\mathbf{A}$ is represented as the continuous, relation-preserving maps from $\mathrm{D}(\mathbf{A})$ to $\underset{\sim}{\mathbf{M}}$. Formally,

$$\begin{aligned} &\mathrm{D}\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{X}} && \mathrm{D} = \boldsymbol{\mathcal{A}}(-, \mathbf{M}) \\ &\mathrm{E}\colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{A}} && \mathrm{E} = \boldsymbol{\mathcal{X}}(-, \underset{\sim}{\mathbf{M}}) \end{aligned}$$

and $\mathbf{A} \simeq \mathrm{ED}(\mathbf{A})$ via $a \mapsto e(a)$.

For example, in a bounded distributive lattice $\mathbf{D}$ is represented by the continuous order-preserving maps from its dual space into $\underset{\sim}{\mathbf{2}} = \langle \{0, 1\}; \mathcal{T}, \leqslant \rangle$. This is isomorphic to the clopen up-sets of the Priestley space of prime filters (ordered by inclusion).

Examples

Boolean algebras:

$\boldsymbol{\mathcal{B}} = \mathbb{ISP}(\mathbf{M})$ where $\mathbf{M} = \langle \{0,1\}; \wedge, \vee, \neg, 0, 1\rangle$

$\underset{\sim}{\mathbf{M}} = \langle \{0,1\}; \mathcal{T}\rangle$ and $\boldsymbol{\mathcal{X}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ = Stone spaces

Bounded distributive lattices:

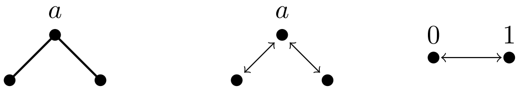
$\boldsymbol{\mathcal{D}} = \mathbb{ISP}(\mathbf{M})$ where $\mathbf{M} = \langle \{0,1\}; \wedge, \vee, 0, 1\rangle$

$\underset{\sim}{\mathbf{M}} = \langle \{0,1\}; \leqslant, \mathcal{T}\rangle$ and $\boldsymbol{\mathcal{X}} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ = Priestley spaces

Kleene algebras:

$\boldsymbol{\mathcal{K}} = \mathbb{ISP}(\mathbf{K})$ where $\mathbf{K} = \langle \{0,a,1\}; \wedge, \vee, \neg, 0, 1\rangle$

$\underset{\sim}{\mathbf{K}} = \langle \{0,a,1\}; R_1, R_2, R_3, \mathcal{T}\rangle$



R_1 R_2 R_3

Examples

Distributive bilattices:

$\boldsymbol{\mathcal{DB}} = \mathbb{ISP}(\mathcal{FOUR})$ where $\mathcal{FOUR} = \langle \{\mathbf{f}, \mathbf{t}, \top, \bot\}; \wedge, \vee, \neg, \mathbf{f}, \mathbf{t} \rangle$

$\underset{\sim}{\mathcal{FOUR}} = \langle \{\mathbf{f}, \mathbf{t}, \top, \bot\}; \leqslant_k, \mathcal{T} \rangle$

We want to develop natural dualities for the quasivarieties $\mathbb{ISP}(J_n)$ for $n \in \omega$.

Theorem (Clark and Davey, 1998)

(NU Duality Theorem, special case) *Let* $\mathbf{M}$ *be a finite lattice-based algebra and* $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$. *Let*

$$\underset{\sim}{\mathrm{M}} = \big(M; \mathbb{S}(\mathbf{M}^2), \mathcal{T} \big)$$

where $\mathcal{T}$ *is the discrete topology* M. *Then* $\underset{\sim}{\mathrm{M}}$ *yields a duality on* $\mathcal{A}$.

Therefore, to find a duality we must study the lattice $\mathbb{S}(\mathbf{J}_n^2)$. We have

$$|\mathbb{S}(\mathbf{J}_0^2)| = 4 \quad \text{and} \quad |\mathbb{S}(\mathbf{J}_1^2)| = 7$$

but

$$|\mathbb{S}(\mathbf{J}_0^2)| = 28 \quad \text{and} \quad |\mathbb{S}(\mathbf{J}_3^2)| = 200$$

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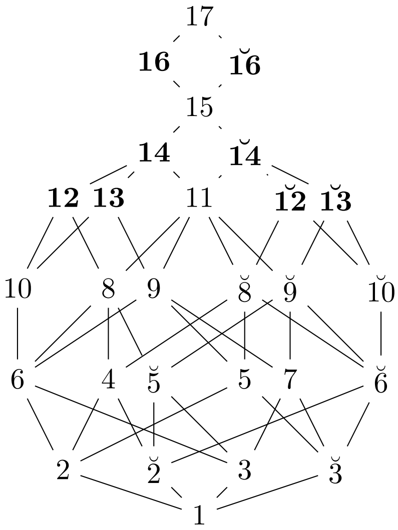

Figure: The algebraic lattice $\mathbb{S}(\mathbf{J}_2^2)$ with its completely meet-irreducible elements shown in bold.

For J_n and $i < n$, consider binary relation

$$S_{n,i} := \{(\top,\top),(\bot,\bot)\} \cup \{\mathbf{f}_{i+1},\dots,\mathbf{f}_n\}^2 \cup \{\mathbf{t}_{i+1},\dots,\mathbf{t}_n\}^2$$
$$\cup \{(\mathbf{f}_j,\mathbf{f}_k) \mid i < j, 0 \leqslant k \leqslant i\} \cup \{(\mathbf{t}_j,\mathbf{t}_k) \mid i < j, 0 \leqslant k \leqslant i\}.$$

Figure: The binary relations $\mathbf{S}_{3,0}, \mathbf{S}_{3,1}$ and $\mathbf{S}_{3,2}$ on J_3 drawn as quasi-orders. We draw x and y in the same block if xRy and yRx.

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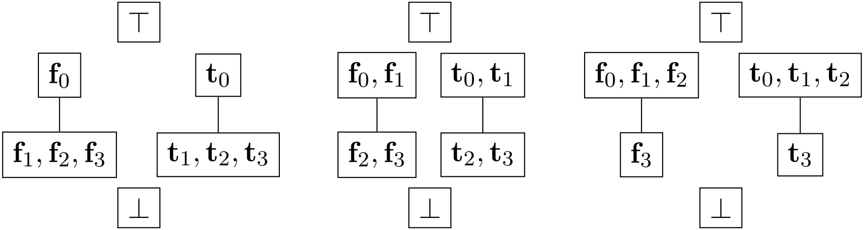

$\mathbf{S}_{3,0}$ $\mathbf{S}_{3,1}$ $\mathbf{S}_{3,2}$

Figure: The binary relations $\mathbf{S}_{3,0}, \mathbf{S}_{3,1}$ and $\mathbf{S}_{3,2}$ on J_3 drawn as quasi-orders. We draw x and y in the same block if xRy and yRx.

Proposition

Consider the bilattice $\mathbf{J}_n$*. For* $i \in \{0, 1, \ldots, n-1\}$ *let* $S_{n,i} \subseteq J_n^2$ *be defined by*

$$
\begin{aligned}
S_{n,i} := \{(\top,\top),(\bot,\bot)\} \cup \{\mathbf{f}_{i+1},\ldots,\mathbf{f}_n\}^2 \cup \{\mathbf{t}_{i+1},\ldots,\mathbf{t}_n\}^2 \\
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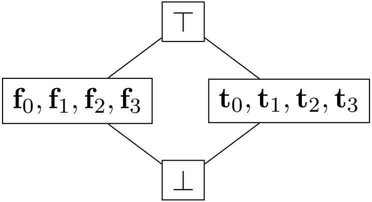
\end{aligned}
$$

Then $S_{n,i}$ *is the universe of a subalgebra of* $\mathbf{J}_n^2$*.*

Proposition

Let S_{nn} be the universe of the subalgebra of $\mathbf{J}_n^2$ generated by the relation $\leqslant_k$ on J_n^2. The relation S_{nn} or its converse must always be included in the dualising structure $\underset{\sim}{\mathbf{J}}_n$.



$\mathbf{S}_{3,3}$

We use a technique called "piggybacking" (developed in different levels of generality by Davey, Werner and Priestley). The method identifies a dualising structure $\underset{\sim}{\mathbf{M}}$ when $\mathbf{M}$ is a finite algebra with a distributive lattice reduct. As observed above, $\mathbf{J}_n$ is always a distributive lattice in its truth order.

Define for $x \neq \mathbf{t}_n$, $w_x : \mathbf{J}_n^\flat \to \underline{\mathbf{2}}$ by

$$w_x(a) = \begin{cases} 1 & \text{if } x \leqslant_t a \\ 0 & \text{otherwise} \end{cases}$$

Now consider

$$W_{x,y} := (w_x, w_y)^{-1}(\leqslant) = \{\, (a,b) \in \mathbf{J}_n^2 \mid w_x(a) \leqslant w_y(b) \,\}$$

The Piggyback Theorem says that we only need the maximal subalgebras contained in the $W_{x,y}$.

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Theorem

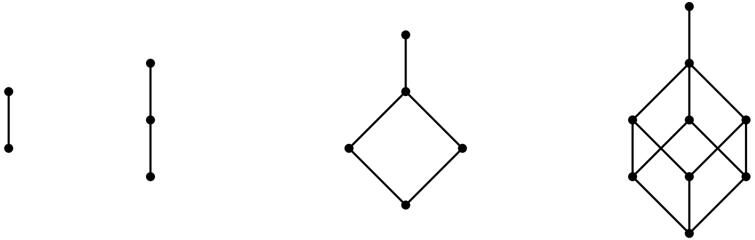
The structure $\underset{\sim}{\mathbf{J}}_n := \langle J_n; S_{n,n}, \ldots, S_{n,0}, \mathcal{T} \rangle$ *yields a duality on the quasivariety* $\mathbb{ISP}(\mathbf{J}_n)$.

A natural duality is called *optimal* if it is not possible to obtain a duality if any of the relations are removed from the dualising structure.

Theorem

The structure $\underset{\sim}{\mathbf{J}}_n := \langle J_n; S_{n,n}, \ldots, S_{n,0}, \mathcal{T}\rangle$ *yields an optimal duality on the quasivariety* $\mathbb{ISP}(\mathbf{J}_n)$*.*

The varieties $\mathbb{HSP}(\mathbf{J}_n)$



$\mathrm{Con}(\mathbf{J}_0)$ $\mathrm{Con}(\mathbf{J}_1)$ $\mathrm{Con}(\mathbf{J}_2)$ $\mathrm{Con}(\mathbf{J}_3)$

Proposition

For each $n \in \omega$ we have $Con(\mathbf{J}_n) \cong (\underline{\mathbf{2}}^n) \oplus 1$.

We will have that $\mathcal{V}(\mathbf{J}_n) = \mathbb{ISP}(\mathbb{H}(\mathbf{J}_n))$ and can then use the theory of multi-sorted natural duality to study dualities for $\mathcal{V}(\mathbf{J}_n)$.

Further work

- Verify that the dualities given above are full dualities.
- Explore the relationship between the natural duality and restricted Priestley duality.
- Investigate adding an implication to the signature.