Bilattices with two chains of truth values

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Introduction Bilattices are algebras with two lattice structures and a unary involutive operation that preserves one lattice order while reversing the other. They are commonly used as algebraic semantics for logics used to model reasoning with incomplete or inconsistent information. Here we present and study a new class of bilattices to be used for modelling situations where there are multiple degrees of both truth and falsehood. The multiple values for *true* and *false* each form a chain, and these chains are unrelated in one of the two lattice orders.

The two sets of lattice operations on a bilattice give rise to two lattice orders. The operations $\otimes$ and $\oplus$ (sometimes interpreted as "consensus" and "gullability") determine the *knowledge order* ($\leqslant_k$) while the operations $\wedge$ and $\vee$ determine the *truth order* ($\leqslant_t$). The unary operation $\neg$ is known as a negation; it reverses the truth order but preserves the knowledge order.

The seven-element bilattice proposed by Ginsberg [6] (often called $\mathcal{SEVEN}$) was proposed as a model for the situation for default logics. In [2], a hierarchy of default bilattices was studied. The hierarchy's smallest member was $\mathcal{FOUR}$ — the original bilattice first proposed by Belnap [1] in the 1970's. The bilattice $\mathcal{FOUR}$ has elements $\mathbf{t}, \mathbf{f}, \top$ and $\bot$, representing, respectively, "true", "false", "contradiction" and "no information".

Aside from the base case $\mathcal{FOUR}$, the bilattices examined here differ from those in [2] in two important ways. Firstly, the truth lattice is distributive while the knowledge lattice is non-distributive. This is a reversal of the situation in [2]. The second difference is that our bilattices do not possess multiple degrees of contradiction; any degree of falsehood combined with any degree of truth by the knowledge join $\oplus$ will result in an interpretation of $\top$.

In addition to introducing the hierarchy of bilattices $\mathbf{J}_n$ ($n \in \omega$), we study the quasivarieties $\mathbb{ISP}(\mathbf{J}_n)$ generated by each of the bilattices. Our approach to studying the quasivarieties is to use natural duality theory [4].

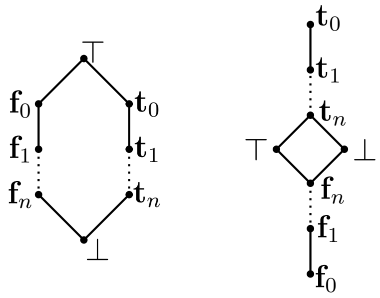

Figure 1: The bilattice $\mathbf{J}_n$ drawn in both its knowledge (left) and truth (right) orders.

The bilattices $\mathbf{J}_n$ The bilattice $\mathbf{J}_n$ has as its underlying set $J_n = \{\bot, \mathbf{f}_0, \mathbf{t}_0, \dots, \mathbf{f}_n, \mathbf{t}_n, \top\}$. The two lattice orders are shown in Fig. 1. The element $\mathbf{f}_0$ has the highest degree of falseness (it is the bottom element of the truth lattice) but it has a maximal degree of non-contradictory information. Similarly, $\mathbf{t}_0$ is one of two most reliable pieces of information; it also has the highest degree of truth. We let $C_n := \{\mathbf{f}_0, \mathbf{t}_0, \dots, \mathbf{f}_n, \mathbf{t}_n\}$. Each element of C_n will appear as a nullary operation in the algebraic signature. The negation operation, $\neg$, is defined by $\neg\mathbf{f}_i = \mathbf{t}_i$, $\neg\mathbf{t}_i = \mathbf{f}_i$, $\neg\top = \top$ and $\neg\bot = \bot$ $(0 \leqslant i \leqslant n)$. Clearly $\neg$ preserves the knowledge order while reversing the truth order. We have $\mathbf{J}_n = \langle J_n; \otimes, \oplus, \wedge, \vee, \neg, C_n \rangle$.

A bilattice is *interlaced* if the truth operations preserve the knowledge order and the knowledge operations preserve the truth order. The bilattices $\mathbf{J}_n$ are not interlaced for $n \geqslant 1$. This can be seen by noting that $\mathbf{f}_0 \leqslant_k \top$ but $\bot \wedge \mathbf{f}_0 = \mathbf{f}_0 \nleqslant_k \mathbf{f}_n = \top \wedge \bot$. A powerful tool in the study of bilattices has been the Product Representation Theorem for interlaced bilattices. (A history of the theorem and details of those who have discovered it in different forms has been written by Davey [5].) As a result, interlaced bilattices (and distributive bilattices, see [3]) have been the subject of much study. This is one of few investigations into non-interlaced bilattices.

Natural duality for $\mathbb{ISP}(\mathbf{J}_n)$ In its most basic form, natural duality is used to study quasivarieties of algebras of the form $\mathbb{ISP}(\mathbf{M})$ where $\mathbf{M}$ is a finite algebra. The challenge is to identify a category of topological spaces, with additional relational structure, that will be dual to $\mathbb{ISP}(\mathbf{M})$. This is done by considering the underlying set M of $\mathbf{M}$ and equipping it with the discrete topology and a set of relations. We will denote this topologised relational structure by $\underset{\sim}{\mathbf{M}}$ and call it the *dualising structure*. Natural duality theory then allows us to study $\mathbb{ISP}(\mathbf{M})$ by looking at the dual category $\mathfrak{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ (isomorphic copies of closed substructures of non-empty products of $\underset{\sim}{\mathbf{M}}$).

The challenge lies in identifying a "suitable" (and small) set of relations on M. For an n-ary relation on M to be "suitable" its underlying set must form a subalgebra of $\mathbf{M}^n$. Since our algebras are lattice-based, we can appeal to the NU Duality Theorem [4, Theorem 2.3.4] and hence only require binary relations in our dualising structure. Identifying the most suitable relations thus amounts to a study of the algebraic lattice $\mathbb{S}(\mathbf{J}_n^2)$.

Proposition 1. *Consider the bilattice $\mathbf{J}_n$. For $i \in \{0, 1, \ldots, n-1\}$ let $S_{n,i} \subseteq J_n^2$ be defined by*

$$\begin{aligned} S_{n,i} := \{(\top,\top),(\bot,\bot)\} \cup \{\mathbf{f}_{i+1},\ldots,\mathbf{f}_n\}^2 \cup \{\mathbf{t}_{i+1},\ldots,\mathbf{t}_n\}^2 \\ \cup \{(\mathbf{f}_j,\mathbf{f}_k) \mid i<j, 0 \leqslant k \leqslant i\} \cup \{(\mathbf{t}_j,\mathbf{t}_k) \mid i<j, 0 \leqslant k \leqslant i\}. \end{aligned}$$

Then $S_{n,i}$ is the universe of a subalgebra of $\mathbf{J}_n^2$.

Proposition 2. *Let S_{nn} be the universe of the subalgebra of $\mathbf{J}_n^2$ generated by the relation $\leqslant_k$ on J_n^2. The relation S_{nn} or its converse must always be included in the dualising structure $\underset{\sim}{\mathbf{J}}_n$.*

The technique used to prove the theorem below is called "piggybacking". The method identifies a dualising structure $\underset{\sim}{\mathbf{M}}$ when $\mathbf{M}$ is a finite algebra with a distributive lattice reduct. As observed above, $\mathbf{J}_n$ is always a distributive lattice in its truth order.

Theorem 3. *The structure $\underset{\sim}{\mathbf{J}}_n := \langle J_n; S_{n,n}, \ldots, S_{n,0}, \mathcal{T}\rangle$ yields a duality on the quasivariety $\mathbb{ISP}(\mathbf{J}_n)$.*

A significant feature of the set of dualising relations in Theorem 3 is the way in which they minimally encode the knowledge order $\leqslant_k$ on $\mathbf{J}_n$.

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