Constructive Canonicity of Inductive Inequalities

Willem Conradie[1] and Alessandra Palmigiano[12]

[1] Department of Pure and Applied Mathematics, University of Johannesburg, South Africa
wconradie@uj.ac.za

[2] Faculty of Technology, Policy and Management, Delft Technical University, The Netherlands
A.Palmigiano@tudelft.nl

Abstract

We prove the canonicity of inductive inequalities in a constructive meta-theory, for classes of logics algebraically captured by varieties of normal and regular lattice expansions. This result encompasses Ghilardi-Meloni's and Suzuki's constructive canonicity results for Sahlqvist formulas and inequalities, and is based on an application of the tools of unified correspondence theory. Specifically, we provide an alternative interpretation of the language of the algorithm ALBA for lattice expansions: nominal and conominal variables are respectively interpreted as closed and open elements of canonical extensions of normal/regular lattice expansions, rather than as completely join-irreducible and meet-irreducible elements of perfect normal/regular lattice expansions. We show the correctness of ALBA with respect to this interpretation. From this fact, the constructive canonicity of the inequalities on which ALBA succeeds follows by a straightforward adaptation of the standard argument. The claimed result then follows as a consequence of the success of ALBA on inductive inequalities.

Canonicity: two approaches. Perhaps the most important uniform methodology for proving completeness for modal logics is the notion of canonicity, which, thanks to duality, can be studied both frame-theoretically and algebraically. The study of canonicity has been extended from classical normal modal logic to its many neighbouring logics, and has given rise to a rich literature. Particularly relevant to the present paper are two general methods for canonicity, pioneered by Sambin and Vaccaro [8] and by Ghilardi and Meloni [4]. Sambin and Vaccaro obtain canonicity for Sahlqvist formulas of classical modal logic in a frame-theoretic setting as a byproduct of correspondence, and their proof strategy has accordingly become known as *canonicity-via-correspondence*.

Ghilardi and Meloni's work [4] shows that canonicity can be meaningfully investigated purely algebraically, in a constructive meta-theory where correspondence is not even defined, in general. Indeed, in [4], the canonical extension construction for certain bi-intuitionistic modal algebras, later applied also to general lattice and poset expansions in [3, 2], is formulated in terms of general filters and ideals, and does not depend on any form of the axiom of choice (such as the existence of 'enough' optimal filter-ideal pairs). Thus, while the constructive canonical extension need not be perfect in the sense of [6], the canonical embedding map, sending the original algebra into its canonical extension, retains the properties of denseness and compactness. These properties make it possible for the authors of [4] to identify a class of constructively canonical inequalities.

Unified Correspondence and Canonicity. The approaches of Sambin-Vaccaro, on the one hand, and Ghilardi-Meloni, on the other, have been very influential, and have contributed to a certain binary divide detectable in the literature between correspondence and canonicity, namely: correspondence being typically done on frames, and canonicity on algebras. Moreover, subsequent algebraic proofs of canonicity have mostly remained restricted to Sahlqvist formulas, rather than considering e.g. the wider class of inductive formulas [5], for which the first proof of algebraic canonicity has appeared only very recently [7]. *Unified correspondence theory* [1], to which the contributions of the present paper belong, bridges this divide in the sense that will be explained below, and by doing this, succeeds in importing Sambin-Vaccaro's proof strategy to the constructive setting of Ghilardi-Meloni, thus providing a conceptual unification of these very different perspectives. For instance, the intermediate step of [4] can be recognized as the equivalent rewriting, independent of the evaluation of proposition variables, pursued in [8].

At the core of unified correspondence is an *algebraic* reformulation of algorithmic *correspondence* theory, with its ensuing algebraic canonicity-via-correspondence argument. This reformulation makes it possible to construe the computation of first-order correspondents in two phases: reduction and translation. Formulas/inequalities are interpreted in the canonical extension $\mathbb{A}^\delta$ of a given algebra $\mathbb{A}$, and a calculus of rules (captured by the ALBA algorithm) is applied to rewrite them into equivalent expressions with no occurring propositional variables, called pure. The pure expressions may, however, contain (non-propositional) variables known as nominals and co-nominals. If successful, achieving pure expressions completes the reduction phase. Pure expressions are already enough to implement Sambin-Vaccaro's canonicity strategy: indeed, the validity of pure expressions under assignments sending propositional variables into $\mathbb{A}$ (identified with the *admissible* assignments on $\mathbb{A}^\delta$) is tantamount to their validity w.r.t. arbitrary assignments on $\mathbb{A}^\delta$, and this establishes the canonicity of the original formula or inequality.

Moving to a constructive setting. In a non-constructive setting, the nominals and co-nominals are interpreted as ranging over the completely join-irreducible or meet-irreducible elements of $\mathbb{A}^\delta$. The soundness of the rewrite rules is based, in part, on the fact that the completely join-irreducible and meet-irreducible elements respectively join-generate and meet-generate the ambient algebra $\mathbb{A}^\delta$. Moreover, in this setting, completely meet- and join-irreducible elements correspond, via discrete duality, to first-order definable subsets of the dual relational semantics. Thus the first-order frame correspondent of the original formula or inequality can be obtained by simply applying the appropriate standard translation to the pure expressions. This is known as the translation phase.

In a constructive setting, the situation just described is changed by the fact that we can no longer rely on the completely join-irreducible and meet-irreducible elements to respectively join-generate and meet-generate $\mathbb{A}^\delta$. However, we may fall back on the closed and open elements of $\mathbb{A}^\delta$ as complete join- and meet-generators, and adjust the interpretations of nominals and co-nominals accordingly. By doing this, the reduction phase remains sound in the non-constructive setting, and still yields canonicity. As expected, however, we lose discrete duality and with it the possibility of translating to the relational semantics.

Contributions. The purpose of the present work is to work out the details and to prove the correctness of the picture sketched in the previous paragraph. We do this in the setting of arbitrary normal or regular lattice expansions. We define the Inductive and Sahlqvist inequalities in the setting of lattices with normal and regular operations. We introduce a constructive version of ALBA, and prove its correctness. We next show that constructive ALBA successfully reduces all inductive and Sahlqvist inequalities, and prove that all inequalities on which constructive ALBA succeeds are constructively canonical. From this our main theorem follows, i.e. that all inductive inequalities are constructively canonical.

References

[1] W. Conradie, S. Ghilardi, and A. Palmigiano. Unified correspondence. In A. Baltag and S. Smets, editors, *Johan van Benthem on Logic and Information Dynamics*, volume 5 of *Outstanding Contributions to Logic*, pages 933–975. Springer International Publishing, 2014.

[2] J. M. Dunn, M. Gehrke, and A. Palmigiano. Canonical extensions and relational completeness of some substructural logics. *Journal Symbolic Logic*, 70(3):713–740, 2005.

[3] M. Gehrke and J. Harding. Bounded lattice expansions. *Journal of Algebra*, 238(1):345–371, 2001.

[4] S. Ghilardi and G. Meloni. Constructive canonicity in non-classical logics. *Annals of Pure and Applied Logic*, 86(1):1–32, 1997.

[5] V. Goranko and D. Vakarelov. Elementary canonical formulae: Extending Sahlqvist's theorem. *Annals of Pure and Applied Logic*, 141(1-2):180–217, 2006.

[6] B. Jónsson and A. Tarski. Boolean algebras with operators. *American Journal of Mathematics*, 74:127–162, 1952.

[7] A. Palmigiano, S. Sourabh, and Z. Zhao. Jónsson-style canonicity for ALBA-inequalities. *Journal of Logic and Computation*, 2015.

[8] G. Sambin and V. Vaccaro. A new proof of Sahlqvist's theorem on modal definability and completeness. *Journal of Symbolic Logic*, 54(3):992–999, 1989.