Algorithmic Correspondence for Many-valued Modal Logic

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Introduction Correspondence theory explores connections between modal languages and first-order languages. Intuitively, two formulas, one from each of these languages, are called each other's *local frame correspondents* if they say the same thing about relational structures at the level of frame validity. Correspondence theory for modal logic is well-developed [9, 6], and various algorithmic approaches [1, 2, 7] are firmly established. We explore correspondence theory between the many-valued modal logics first introduced by Fitting [3, 4], and many-valued first-order logics. In particular, we investigate how the correspondence algorithm ALBA [2] which computes first-order frame correspondents for certain modal formulas, can be adapted to the many-valued setting. We define the appropriate classes of inductive and Sahlqvist formulas in this setting, and prove that the modified ALBA algorithm successfully computes first-order frame correspondents for all of them. We compare our results to related results to be found in the literature, including [5, 8].

Many-valued Modal Logics Let $L_{\mathbb{A}}$ be the basic many-valued modal language with formulas recursively defined by

$$\varphi := \mathbf{t} \mid p \mid p \vee q \mid p \wedge q \mid p \rightarrow q \mid \Diamond p \mid \Box p,$$

where the p comes from a denumerably infinite set of proposition letters $Prop$, and $\mathbf{t}$ is a truth value constant for and for each $t \in A$. In particular $\mathbf{0}$ (*falsum*) and $\mathbf{1}$ (*verum*) are truth value constants in $L_{\mathbb{A}}$. Note also that negation can be defined as $\neg p = p \rightarrow \mathbf{0}$.

Following Fitting [3, 4] we will consider two families of many-valued modal logics over a perfect (not necessarily finite) Heyting algebra $\mathbb{A}$. The models of the first family of logics under consideration have crips (binary) accessability relations and a many-valued valuation; while the models of the second family also have many-valued accessability relations. That is, given a nonempty universe W, an

$\mathbb{A}$-valued frame (or, an $\mathbb{A}$-frame) is a triple $\mathfrak{F} = (W, D, B)$, where D and B can be either binary accessability relations on W (the crisp family) or functions such that $D : (W \times W) \to A$ and $B : (W \times W) \to A$ (the many-valued family). In either case, an $\mathbb{A}$-valued model (or, an $\mathbb{A}$-model) is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is an $\mathbb{A}$-frame and V is a function such that $V : (\mathit{Prop} \times W) \to A$. It should be clear that the first family of logics is a special case of the second.

The first-order language $L_{\mathbb{A}}^{FO}$ in which we wish to obtain correspondents for certain formulas from $L_{\mathbb{A}}$ includes binary relation symbols B and D, unary many-valued predicate symbols $P_1, P_2, \ldots$, as well as constant symbols corresponding to each truth-value in the truth-value space $\mathbb{A}$. Validity and truth with respect to $\mathbb{A}$ are defined as follows.

Definition 1. *Let $a \in A$. A formula ϕ of $L_{\mathbb{A}}$ is* a-valid *in an $\mathbb{A}$-frame $\mathfrak{F}$ at $w \in W$ (notation: $\mathfrak{F}, w \Vdash_a \phi$) if $V(\phi, w) \geq a$ for all valuations V on $\mathfrak{F}$.*

Definition 2. *Let $a \in A$. We say that a formula β of $L_{\mathbb{A}}^{FO}$ is* a-true *in an interpretation $\mathfrak{I}$ (notation: $\mathfrak{I} \vDash_a \beta$) if $v(\beta) \geq a$ for all assignments v in $\mathfrak{I}$.*

The notion of a local frame correspondent is extended to the many-valued case in the following way:

Definition 3. *Let ϕ be a formula of $L_{\mathbb{A}}$, $\alpha(x)$ a formula of $L_{\mathbb{A}}^{FO}$ in one free variable x, and $a \in A$ such that $a \neq 0$. We say that ϕ and α are* local frame a-correspondents *if*

$$\mathfrak{F}, w \Vdash_a \phi \Longleftrightarrow \mathfrak{F} \vDash_a \alpha[x := w]$$

for every $\mathbb{A}$-valued frame $\mathfrak{F}$ and all points w in $\mathfrak{F}$.

Many-valued ALBA. The ALBA algorithm applies a calculus of rewrite rules in order to reduce formulas of propositional non-classical logics to pure formulas in an extended hybrid language. These hybrid formulas are equivalently translatable into first-order formulas when interpreted over the relational semantics. The soundness of the rewrite rules is based on the order-theoretic properties of the interpretation of the connectives in the algebraic semantics. Therefore, in order to formulate a suitable version of ALBA in the many-valued setting, we first need to establish a suitable duality between the Kripke semantics for our two families of many-valued modal logics and the appropriate algebraic semantics, which turns out to be classes of Heyting algebras. These Heyting algebras, called $\mathbb{A}$-valued modal algebras (or $\mathbb{A}$-MAs), are perfect and, for example, in the case where we have many-valued accessibility relations, they are defined as below.

Definition 4. *Let* $\mathbb{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ *and* $W \neq \emptyset$. *An* $\mathbb{A}$*-MA is a power algebra* $\mathbb{G} = (\mathbb{A}^W, \Diamond, \Box, \{\mathbf{t}\}_{t \in A})$ *such that for all* $x \in W$ *and for all* $f \in A^W$*:*

$$(\Diamond f)(x) \leq \bigvee\{f(y) \wedge \bigwedge\{g(y) \rightarrow (\Diamond g)(x) \mid \forall g \in A^W\} \mid y \in W\}$$
$$(\Box f)(x) \geq \bigwedge\{\bigwedge\{(\Box g)(x) \rightarrow g(y) \mid \forall g \in A^W\} \rightarrow f(y) \mid y \in W\}$$

We prove duality between $\mathbb{A}$-frames and $\mathbb{A}$-MAs in both the crisp and many-valued cases.

We prove the soundness of ALBA's rewrite rules for the many-valued setting.

Theorem 1. *If ALBA succeeds in reducing an inequality* $\phi \leq \psi$ *and yields a first-order formula* $ALBA(\phi \leq \psi)(x)$, *then* $\mathfrak{F}, w \Vdash \phi \leq \psi$ *iff* $\mathfrak{F} \vDash ALBA(\phi \leq \psi)[x := w]$.

Theorem 2. *ALBA successfully reduces all inductive (and hence Sahlqvist) formulas in the language* $L_{\mathbb{A}}$.

Corallory 1. *All inductive (and hence Sahlqvist)* $L_{\mathbb{A}}$ *formulas have local frame correspondents in* $L_{\mathbb{A}}^{FO}$.

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