Constructive Canonicity of Inductive Inequalities

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Canonical Extensions

Definition

Let $\mathbb{A}$ be a (bounded) sublattice of a complete lattice $\mathbb{A}'$.

1. $\mathbb{A}$ is dense in $\mathbb{A}'$ if every element of $\mathbb{A}'$ can be expressed both as a join of meets and as a meet of joins of elements from $\mathbb{A}$.
2. $\mathbb{A}$ is compact in $\mathbb{A}'$ if, for all $S, T \subseteq \mathbb{A}'$, if $\bigvee S \leq \bigwedge T$ then $\bigvee S' \leq \bigwedge T'$ for some finite $S' \subseteq S$ and $T' \subseteq T$.
3. The canonical extension of a lattice $\mathbb{A}$ is a complete lattice $\mathbb{A}^\delta$ containing $\mathbb{A}$ as a dense and compact sublattice.

Closed elements $K(\mathbb{A}^\delta)$: Meet closure of $\mathbb{A}$ in $\mathbb{A}^\delta$.
Open elements $O(\mathbb{A}^\delta)$: Join closure of $\mathbb{A}$ in $\mathbb{A}^\delta$.

Canonical Extensions

Theorem (Propositions 2.6 and 2.7 in [1])

The canonical extension of a bounded lattice $\mathbb{A}$ exists and is unique up to any isomorphism fixing $\mathbb{A}$.

In the presence of AC, $\mathbb{A}^\delta$ is:

- completely join-generated by $J^\infty(\mathbb{A}^\delta)$
- completely meet-generated by $M^\infty(\mathbb{A}^\delta)$

Extension of unary maps

Definition

For every unary, order-preserving operation $f : \mathbb{A} \to \mathbb{A}$, the *$\sigma$-extension* of f is defined firstly by declaring, for every $k \in K(\mathbb{A}^\delta)$,

$$f^\sigma(k) := \bigwedge\{f(a) \mid a \in \mathbb{A} \text{ and } k \leq a\},$$

and then, for every $u \in \mathbb{A}^\delta$,

$$f^\sigma(u) := \bigvee\{f^\sigma(k) \mid k \in K(\mathbb{A}^\delta) \text{ and } k \leq u\}.$$

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The π-*extension* of f is defined firstly by declaring, for every $o \in O(\mathbb{A}^\delta)$,

$$f^\pi(o) := \bigvee\{f(a) \mid a \in \mathbb{A} \text{ and } a \leq o\},$$

and then, for every $u \in \mathbb{A}^\delta$,

$$f^\pi(u) := \bigwedge\{f^\pi(o) \mid o \in O(\mathbb{A}^\delta) \text{ and } u \leq o\}.$$

Extension of n-ary maps

- $f : \mathbb{A}^{\varepsilon} \to \mathbb{A}$ preserves (finite, possibly empty) $\vee$s coordinate-wise.
- $g : \mathbb{A}^{\varepsilon} \to \mathbb{A}$ preserves (finite, possibly empty) $\wedge$s coordinate-wise.

Definition

- $f^{\sigma}(\overline{k}) := \bigwedge\{f(\overline{a}) \mid \overline{a} \in (\mathbb{A}^{\delta})^{\varepsilon_f} \text{ and } \overline{k} \leq^{\varepsilon_f} \overline{a}\}$
- $f^{\sigma}(\overline{u}) := \bigvee\{f^{\sigma}(\overline{k}) \mid \overline{k} \in K((\mathbb{A}^{\delta})^{\varepsilon_f}) \text{ and } \overline{k} \leq^{\varepsilon_f} \overline{u}\}$
- $g^{\pi}(\overline{o}) := \bigvee\{g(\overline{a}) \mid \overline{a} \in (\mathbb{A}^{\delta})^{\varepsilon_g} \text{ and } \overline{a} \leq^{\varepsilon_g} \overline{o}\}$
- $g^{\pi}(\overline{v}) := \bigwedge\{g^{\pi}(\overline{o}) \mid \overline{o} \in O((\mathbb{A}^{\delta})^{\varepsilon_g}) \text{ and } \overline{v} \leq^{\varepsilon_g} \overline{o}\}$.

Canonical Extension of an LE

Definition

The canonical extension of an $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A} = (L, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ is the $\mathcal{L}_{\mathrm{LE}}$-algebra $\mathbb{A}^{\delta} := (L^{\delta}, \mathcal{F}^{\mathbb{A}^{\delta}}, \mathcal{G}^{\mathbb{A}^{\delta}})$ such that $f^{\mathbb{A}^{\delta}}$ and $g^{\mathbb{A}^{\delta}}$ are defined as the σ-extension of $f^{\mathbb{A}}$ and as the π-extension of $g^{\mathbb{A}}$ respectively, for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

- $\mathbb{A}^{\delta}$ is quasi-perfect.
- In the presence of AC, $\mathbb{A}^{\delta}$ is perfect.

Canonical equations and inequalities

Definition

Equation $\varphi = \psi$ is canonical if

$$\mathbb{A} \models \varphi = \psi \quad \text{implies} \quad \mathbb{A}^{\delta} \models \varphi = \psi$$

Theorem (optional)

All Sahlqvist formulas are canonical.

Approach 1: Canonicity via Correspondence

Sambin and Vacarro (1989) [2], prove that all Sahlqvist inequalities are canonical.

Suppose $\varphi \to \psi$ has F.O.-frame $\alpha(x)$. Then

$$
\begin{aligned}
& (\mathfrak{F}, \mathbb{A}) \Vdash \varphi \leq \psi && (1)\\
\text{iff} \quad & (\mathfrak{F}, \mathbb{A}) \models \alpha(x) && (2)\\
\text{iff} \quad & (\mathfrak{F}, \mathbb{A}^{\delta}) \models \alpha(x) && (3)\\
\text{iff} \quad & (\mathfrak{F}, \mathbb{A}^{\delta}) \Vdash \varphi \leq \psi && (4)
\end{aligned}
$$

Same approach followed, but in purely modal environment, by SQEMA (C., Goranko & Vakarelov, [3]) and ALBA (C., Palmigiano, [4]).

Approach 2: Algebraic

Ghilardi and Meloni (1997) [5]:

- Constructive; correspondence not defined in general.

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- Validity lifted from $\mathbb{A}$ to $\mathbb{A}^\delta$ in 2 steps
  1. From elements of $\mathbb{A}$ to $K(\mathbb{A}^\delta)$ and $O(\mathbb{A}^\delta)$,
  2. then from $K(\mathbb{A}^\delta)$ and $O(\mathbb{A}^\delta)$ to arbitrary elements of $\mathbb{A}^\delta$.

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  1. From elements of $\mathbb{A}$ to $K(\mathbb{A}^\delta)$ and $O(\mathbb{A}^\delta)$,
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Strategy similar to Jónsson 1994 [6], but independent.

Canonicity via correspondence: the ALBA way

ALBA (C and Palmigiano 2012 [4]) is a calculus of rewrite rules, that instantiates the following argument:

$$
\begin{array}{ccccc}
\mathbb{A} \models \varphi \leq \psi & & & & \mathbb{A}^\delta \models \varphi \leq \psi \\
\Updownarrow & & & & \\
\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi & & & & \Updownarrow \\
\Updownarrow & & & & \\
\mathbb{A}^\delta \models_{\mathbb{A}} \text{ALBA}(\varphi \leq \psi) & & \Leftrightarrow & & \mathbb{A}^\delta \models \text{ALBA}(\varphi \leq \psi)
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e.g.

Example

$$
\begin{array}{lcl}
\mathbb{A} \models \Diamond\Box p \leq \Box\Diamond p & & \mathbb{A}^\delta \models \Diamond\Box p \leq \Box\Diamond p \\
\Updownarrow & & \\
\mathbb{A}^\delta \models_{\mathbb{A}} \Diamond\Box p \leq \Box\Diamond p & & \Updownarrow \\
\Updownarrow & & \\
\mathbb{A}^\delta \models_{\mathbb{A}} \Diamond\mathbf{j} \leq \Box\Diamond\blacklozenge\mathbf{j} & \Leftrightarrow & \mathbb{A}^\delta \models \Diamond\mathbf{j} \leq \Box\Diamond\blacklozenge\mathbf{j}
\end{array}
$$

Soundness of rules: problems in the constructive setting

e.g. $\Diamond$-approximation rule:

$$\frac{C, \quad \mathbf{i} \leq \Diamond\alpha \quad \Rightarrow \quad D}{C, \quad \mathbf{j} \leq \alpha, \quad \mathbf{i} \leq \Diamond\mathbf{j} \quad \Rightarrow \quad D}$$

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Soundness depends on

- complete $\vee$-preservation of $\Diamond^{\mathbb{A}^\delta}$
- complete $\vee$-generation of $\mathbb{A}^\delta$ by $J^\infty(\mathbb{A}^\delta)$
- all $J^\infty(\mathbb{A}^\delta)$ being completely join-prime

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---

In the constructive and non-distributive setting, this can be replaced with:

$$\frac{S \quad \Rightarrow \quad \varphi'(\gamma/!x) \leq \psi}{S \cup \{\mathbf{j} \leq \gamma\} \quad \Rightarrow \quad \varphi'(\mathbf{j}/!x) \leq \psi)}$$

with

- $+x \prec +\varphi'(!x)$
- branch of $+\varphi'(!x)$ starting at $+x$ being Syntactically Additive Coordinatewis (SAC)
- etc.

Constructive ALBA (1)

Applies to logics with algebraic semantics based on arbitrary normal/regular lattice expansions.

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Approximation rules.

e.g. Left-negative approximation rule.

$$\frac{(S,\ \ \varphi'(\gamma/!x) \leq \psi)}{(S \cup \{\gamma \leq \mathbf{m}\},\ \ \varphi'(\mathbf{m}/!x) \leq \psi)}\ (L^{-}A)$$

with

- $-x \prec +\varphi'(!x)$
- the branch of $+\varphi'(!x)$ starting at $-x$ being SAC
- etc.

Constructive ALBA (2)

Residuation rules.

For normal connectives:

$$\frac{f(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_{n_f}) \leq \psi}{\varphi_i \leq f_i^\sharp(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_f})} \; \varepsilon_f(i) = 1 \qquad \frac{f(\varphi_1, \ldots, \varphi_i, \ldots, \varphi_{n_f}) \leq \psi}{f_i^\sharp(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_f}) \leq \varphi_i} \; \varepsilon_f(i) = \partial$$

Residuation rules.

For normal connectives:

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For regular connectives:

$$\frac{f(\varphi)\leq\psi}{f(\bot)\leq\psi \quad \varphi\leq\blacksquare_f\psi}\ (\text{if } \varepsilon_f=1) \qquad \frac{\varphi\leq g(\psi)}{\varphi\leq g(\top) \quad \blacklozenge_g\varphi\leq\psi}\ (\text{if } \varepsilon_g=1)$$

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For normal connectives:

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Ackermann rules

$$\frac{(\{\alpha_i \leq p\} \cup \{\beta_j(p) \leq \gamma_j(p)\},\ \ \text{Ineq})}{(\{\beta_j(\bigvee \alpha_i) \leq \gamma_j(\bigvee)\},\ \ \text{Ineq})}\ (RAR)$$

where:

- p does not occur in $\alpha_1,\ldots,\alpha_n$ or in Ineq,
- $\beta_1(p),\ldots,\beta_m(p)$ are positive in p, and
- $\gamma_1(p),\ldots,\gamma_m(p)$ are negative in p.

Constructive Canonicity of Inductive and Sahlqvist Formulas

Inductive inequalities.

- Goranko + Vakarelov 2006 [7].
- Essentially subsumes Sahlqvist Formulas.
- General purpose definitions for logics with algebraic semantics based on normal/regular lattice expansion (C. and Palmigiano [4] and [8])

Example

K-axiom: $(\Box(p \to q) \wedge \Box p) \to \Box q$

Frege axiom: $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

Theorem

All inductive inequalities are successfully reducible using rules of Constructive ALBA.

Corollary

All inductive inequalities are constructively canonical.

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