Completeness properties in protoalgebraic logics*

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The papers [1, 3] propose a new approach to abstract algebraic logic in which, instead of the usual equivalence-based classification of logical systems leading to the Leibniz hierarchy of protoalgebraic logics (see [4]), we present an alternative setting based on implication connectives. By studying the properties of implications (understood as generalized connectives defined by sets of formulae in two variables and, possibly, with parameters), we have obtained a refinement of the Leibniz hierarchy with several new classes of *weakly p-implicational logics*. Moreover, we have shown that implications define an order relation in the semantical counterpart of these logics, i.e. in their reduced matrix models.

More precisely, given a propositional logic L (i.e. a structural consequence relation) in a countable language $\mathcal{L}$, let $\Rightarrow(p, q, \overrightarrow{r}) \subseteq Fm_{\mathcal{L}}$ be a set of formulae in two variables (and possibly with some more variables as parameters). We say that $\Rightarrow$ is a *weak p-implication* in L if:[1,2]

(R)	$\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$
(MP)	$\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$
(T)	$\varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi$
(sCng)	$\varphi \Leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \Leftrightarrow c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n)$ for each $\langle c, n\rangle \in \mathcal{L}$ and each $i < n$.

We will also need to make a use of the so-called 'lattice disjunction' i.e., a connective obeying the following conditions:

($\vee$1)	$\vdash_{\mathrm{L}} \varphi \Rightarrow \psi \vee \varphi$
($\vee$2)	$\vdash_{\mathrm{L}} \varphi \Rightarrow \psi \vee \psi$
($\vee$3)	$\varphi \Rightarrow \chi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \vee \psi \Rightarrow \chi$
PCP	if $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$, then $\Gamma, \varphi \vee \psi \vdash_{\mathrm{L}} \chi$

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[1]We write '$\varphi \Rightarrow \psi$' for the set $\{\delta(\varphi, \psi, \alpha_1, \dots, \alpha_n) \mid \delta(p, q, r_1, \dots, r_n) \in \Rightarrow(p, q, \overrightarrow{r})$ and $\alpha_1, \dots, \alpha_n \in Fm_{\mathcal{L}}\}$ and '$\varphi \Leftrightarrow \psi$' for the set $(\varphi \Rightarrow \psi) \cup (\psi \Rightarrow \varphi)$.

[2]The conditions imply that $\Leftrightarrow$ is a parameterized equivalence and, hence, any logic with a weak p-implication is protoalgebraic. Conversely, each protoalgebraic logic has a weak p-implication.

The class of reduced matrix models of L is denoted as $\mathbf{MOD}^*(\mathrm{L})$. For any matrix $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$, the implication defines an order relation as: $a \leq b$ iff $a \Rightarrow^{\boldsymbol{A}} b \subseteq F$, for each $a, b \in A$.

Although $\mathbf{MOD}^*(\mathrm{L})$ already gives a complete semantics for the logic, it is common to consider meaningful subclasses of reduced models which may provide stronger completeness theorems. A matrix $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ is *relatively (finitely) subdirectly irreducible in* $\mathbf{MOD}^*(\mathrm{L})$, in symbols $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ ($\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$), if it cannot be decomposed as a non-trivial subdirect product of an arbitrary (finite non-empty) family of matrices from $\mathbf{MOD}^*(\mathrm{L})$. Note that in lattice-disjunctive logics (i.e., logics with some lattice-disjunction) a reduced model is in $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ iff its filter is $\vee$-prime. The class $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ gives a complete semantics whenever L is finitary.

Let L be a protoalgebraic logic and $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. We say that L has the *strong* $\mathbb{K}$*-completeness*, S$\mathbb{K}$C for short, when for every set of formulae $\Gamma \cup \{\varphi\}$ we have: $\Gamma \vdash_{\mathrm{L}} \varphi$ if, and only if, $\Gamma \models_{\mathbb{K}} \varphi$. We add the prefix 'finite' (or drop 'strong') if the equivalence holds for all finite Γs (or empty Γ respectively).

Of course, the S$\mathbb{K}$C implies the FS$\mathbb{K}$C, and the FS$\mathbb{K}$C implies the $\mathbb{K}$C. The aim of this talk is to prove characterizations of these properties that will allow, for particular choices of logics and classes of reduced models, either to show or to falsify the corresponding completeness properties, and to prove relationships between completeness properties w.r.t. different matricial semantics. In this abstract, as a sample, we present three particular results. The first one is a simple consequence of model-theoretic considerations and is implicit in the literature. The other two utilize the presence of a lattice disjunction.

Theorem 1: *Let* L *be a protoalgebraic logic and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. *Then:*

1. L *has the* $\mathbb{K}$C *if, and only if,* $\mathbf{MOD}^*(\mathrm{L}) \subseteq \mathbf{HSP}(\mathbb{K})$.
2. L *has the* FS$\mathbb{K}$C *if, and only if,* $\mathbf{MOD}^*(\mathrm{L}) \subseteq \mathbf{ISPP}_{\mathrm{U}}(\mathbb{K})$.
3. L *has the* S$\mathbb{K}$C *if, and only if,* $\mathbf{MOD}^*(\mathrm{L}) \subseteq \mathbf{ISP}_{\sigma\text{-}f}(\mathbb{K})$.

Theorem 2: *Let* L *be finitary lattice-disjunctive protoalgebraic logic. Then the following are equivalent:*

1. L *has the* S$\mathbb{K}$C.
2. *Every non-trivial countable matrix from* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ *is embeddable into some member of* $\mathbb{K}$.
3. *Every countable matrix from* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ *is embeddable into some member of* $\mathbb{K}$.

Theorem 3: *Let* L *be finitary lattice-disjunctive protoalgebraic logic in a finite language and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$. *Then the following are equivalent:*

1. L *has the* FS$\mathbb{K}$C.
2. *Every countable matrix from* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ *can be partially embedded into some member of* $\mathbb{K}$.
3. *Every countable matrix from* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ *can be partially embedded into some member of* $\mathbb{K}$.

References

[1] Cintula, Petr, and Carles Noguera, 'Implicational (semilinear) logics I: A new hierarchy', *Archive for Mathematical Logic*, 49 (2010), 4, 417–446.

[2] Cintula, Petr, and Carles Noguera, 'The Proof by Cases Property and its Variants in Structural Consequence Relations', *Studia Logica*, 101 (2013), 4, 713–747.

[3] Cintula, Petr, and Carles Noguera, 'Implicational (semilinear) logics II: additional connectives and characterizations of semilinearity', *Archive for Mathematical Logic*, in press.

[4] Czelakowski, Janusz, *Protoalgebraic Logics*, vol. 10 of *Trends in Logic*, Kluwer, Dordrecht, 2001.