Pavelka-style complete fuzzy logics

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Setting up the stage

Basic notions:

- $\mathcal{L}$ is a propositional language expanding that of MTL
- $\boldsymbol{A}$ is a standard $\mathcal{L}$-algebra
  i.e. $\boldsymbol{A}$ has domain [0,1], usual order, and subuniverse $[0, 1] \cap \mathrm{Q}$
- $\models_{\boldsymbol{A}}$ is the consequence relation given by $\boldsymbol{A}$
- $\mathrm{L}_{\boldsymbol{A}}$ as the finitary companion of $\models_{\boldsymbol{A}}$
  i.e., the largest finitary logic contained in $\models_{\boldsymbol{A}}$

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Enter the rational truth constants:

- $\mathcal{LR}$ is expansion of $\mathcal{L}$ by nullary connectives $\{\overline{r} \mid r \in [0,1] \cap \mathrm{Q}\}$.
- $\boldsymbol{A}^{\mathrm{Q}}$ is the $\mathcal{LR}$-expansion of $\boldsymbol{A}$ by setting $\overline{r}^{\boldsymbol{A}^{\mathrm{Q}}} = r$

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- $\mathrm{QL}_{\boldsymbol{A}}$: the expansion of $\mathrm{L}_{\boldsymbol{A}}$ by the so-called book-keeping axioms:

$$c(\overline{r}_1, \ldots \overline{r}_n) \leftrightarrow \overline{c^{\boldsymbol{A}}(r_1, \ldots, r_n)}$$

for each n-ary connective c in $\mathcal{L}$.

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A rational expansion L of $\mathrm{L}_{\boldsymbol{A}}$ is any logic between $\mathrm{QL}_{\boldsymbol{A}}$ and $\models_{\boldsymbol{A}^{\mathrm{Q}}}$

Logics with rational truth constants — relationships

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- $\models_{A^{\mathrm{Q}}}$, i.e., the logic of A^{Q}
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- If $\mathrm{L}_{\boldsymbol{A}}$ expands the Łukasiewicz logic, then $\mathrm{L}_{\boldsymbol{A}^{\mathrm{Q}}} = \mathrm{QL}_{\boldsymbol{A}}$
  i.e., there is only one finitary rational expansion L of $\mathrm{L}_{\boldsymbol{A}}$
- In general there could be uncountably many different such expansions, e.g., for $\mathrm{L}_{\boldsymbol{A}}$ being the Gödel–Dummett logic

Pavelka-style completeness - PSC

A rational expansion L of $\mathrm{L}_{\boldsymbol{A}}$ enjoys PSC if for every theory $T \cup \{\varphi\}$:

$$\sup\{r \mid T \vdash_{\mathrm{L}} \bar{r} \to \varphi\} = \inf\{e(\varphi) \mid e \text{ is an } \boldsymbol{A}^{\mathrm{Q}}\text{-model of } T\}$$

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Proposition

The logic $\models_{A^{\mathrm{Q}}}$

- *enjoys the PSC*
- *is extension of any rational expansion of L_A with PSC by*

$$\{\bar{r} \to \varphi \mid r < 1\} \vdash \varphi \qquad \text{((PIR))}$$

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Proposition

Assume that $\boldsymbol{A}$ is polar. Then

$$\mathrm{QL}_{\boldsymbol{A}} \textit{ enjoys the PSC iff } c^{\boldsymbol{A}} \textit{ is continuous for each } c \in \mathcal{L}.$$

In presence of discontinuity no finitary rational expansion of $\mathrm{L}_{\boldsymbol{A}}$ enjoys the PSC.

Summary of the 'continuous' case

Corollary

Assume that A *is polar and* c^{A} *is continuous for each* $c \in \mathcal{L}$*. Then*

- L_{A} *expands Łukasiewicz logic*
- $\mathrm{QL}_{A} = \mathrm{L}_{A^{\mathrm{Q}}}$ *and enjoys Pavelka-style completeness and finite strong standard completeness*
- $\models_{A^{\mathrm{Q}}}$ *is an extension of* QL_{A} *by infinitary rule*

$$\{\bar{r} \to \varphi \mid r < 1\} \vdash \varphi \qquad ((\mathrm{PIR}))$$

I.e., $\mathrm{QL}_{A} + (\mathrm{PIR})$ *enjoys strong standard completeness*

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Assume that A is polar; consider binary connective c antitone in the first argument and monotone in the second one.

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Lemma

Let L *a rational expansion of* $\mathrm{L}_{\boldsymbol{A}}$ *with PSC. Then for any* $x, y \in [0,1]$ *and rational* $r < c^{\boldsymbol{A}}(x,y)$. *Then*

- $\{\varphi \to \overline{r}_1, \overline{r}_2 \to \psi \mid r_1 > x$ *and* $r_2 < y\} \vdash_{\mathrm{L}} \overline{r} \to c(\varphi, \psi)$

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and thus for each $r < c^{\boldsymbol{A}}(x,y)$ we must have $T \vdash_{\mathrm{L}} \overline{r} \to c(\varphi, \psi)$.

How to deal with the discontinuities? — 2

Lemma

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- $\{\varphi \to \bar{r}_1, \bar{r}_2 \to \psi \mid r_1 > x \text{ and } r_2 < y\} \vdash_{\mathrm{L}} \bar{r} \to c(\varphi, \psi)$
- $\{\bar{r}_1 \to \varphi, \psi \to \bar{r}_2 \mid r_1 < x \text{ and } r_2 > y\} \vdash_{\mathrm{L}} c(\varphi, \psi) \to \bar{s}$

How to deal with the discontinuities? — 3

Theorem

Let L *be a semilinear rational expansion of a polar logic* $\mathrm{L}_{\boldsymbol{A}}$*. Then the following are equivalent*

- L *enjoys the Pavelka-style completeness*
- L *proves rules* $\overline{r} \vdash \overline{0}$*, for each* $r < 1$ *and rules*

$$\{\overline{r}_i \to^{\hat{c}(i)} \varphi_i \mid r_i <^{\hat{c}(i)} x_i \text{ and } i \leq n\} \vdash_{\mathrm{L}} \overline{r} \to c(\varphi_1, \dots, \varphi_n)$$
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for each

- n*-ary* $c \in \mathcal{L}$ *such that* $c^{\boldsymbol{A}}$ *is non-continuous in* $\langle x_1, \dots, x_n \rangle$
- *and rational* r, s *such that* $r < c^{\boldsymbol{A}}(x_1, \dots, x_n)$ *and* $s > c^{\boldsymbol{A}}(x_1, \dots, x_n)$

If $\boldsymbol{A}$ is *polar* then each n-ary $c \in \mathcal{L}$ has a polarity, i.e., is sequence $\hat{c} \in \{+1, -1\}^n$ such that for each $i \leq n$ the operation $c^{\boldsymbol{A}}$ is monotone in the i-th argument whenever $\hat{c}(i) = +1$ and antitone otherwise.

How to deal with the discontinuities? — Example 1

Recall that in product logic $\to$ in non-continuous in point $\langle 0, 0\rangle$.

Thus the infinitary rules we need to consider look like:

- $\{\varphi \to \bar{r}_1, \bar{r}_2 \to \psi \mid r_1 > 0$ and $r_2 < 0\} \vdash \bar{r} \to (\varphi \to \psi)$
- $\{\bar{r}_1 \to \varphi, \psi \to \bar{r}_2 \mid r_1 < 0$ and $r_2 > 0\} \vdash (\varphi \to \psi) \to \bar{s}$.

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for each $r < 1$ and $s > 1$. Which simplifies (for each $r < 1$) to:

$$\{\varphi \to \overline{r}_1 \mid r_1 > 0\} \vdash \overline{r} \to (\varphi \to \psi)$$

and can be equivalently replaced (for each $0 < r < 1$) by

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Recall that Baaz–Monteiro $\triangle$ in non-continuous in point $\langle 1 \rangle$.

Thus the infinitary rules we need to consider look like:

- $\{\bar{r}_1 \to \varphi \mid r_1 < 1\} \vdash \bar{r} \to \triangle\varphi$
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Problem: in presence of discontinuity no finitary rational expansion of $\mathrm{L}_{\boldsymbol{A}}$ enjoys the PSC.

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Let L be a semilinear rational expansion of a polar logic $\mathrm{L}_{\boldsymbol{A}}$. Then the following are equivalent

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It is hard to prove semilinearity in infinitary logics

Semilinearity in our framework

Theorem

Let L *be an expansion of* MTL *with a countable axiomatic system. TFAE:*

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- $\vee$ *is a strong disjunction, i.e., for each* $\Gamma, \Delta, \Xi, \varphi$ *we have:*

$$\frac{\Gamma, \Delta \vdash_{\mathrm{L}} \varphi \qquad \Gamma, \Xi \vdash_{\mathrm{L}} \varphi}{\Gamma, \{\delta \vee \chi \mid \delta \in \Delta, \chi \in \Xi\} \vdash_{\mathrm{L}} \varphi}$$

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- *For each additional rule of* L *of the form* $\Gamma \vdash \varphi$ *we have:*

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(a consequence of new general result stating that sPCP implies PEP)

The main theorem

Theorem

Let $\boldsymbol{A}$ be a standard polar algebra with countably many points of non-continuity and L the expansion of $\mathrm{QL}_{\boldsymbol{A}}$ by rules $\bar{r} \vdash \bar{0}$, for $r < 1$ and

$$\{(\bar{r}_i \to^{\hat{c}(i)} \varphi_i) \vee \chi \mid r_i <^{\hat{c}(i)} x_i \text{ and } i \leq n\} \vdash_{\mathrm{L}} (\bar{r} \to c(\varphi_1, \dots, \varphi_n)) \vee \chi$$
$$\{(\varphi_i \to^{\hat{c}(i)} \bar{r}_i) \vee \chi \mid x_i <^{\hat{c}(i)} r_i \text{ and } i \leq n\} \vdash_{\mathrm{L}} (c(\varphi_1, \dots, \varphi_n) \to \bar{s}) \vee \chi$$

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Then L the least semilinear rational expansion of a polar logic $\mathrm{L}_{\boldsymbol{A}}$ enjoying the Pavelka-style completeness. Furthermore $\models_{\boldsymbol{A}^{\mathrm{Q}}}$ is its extension by the rule (PIR).

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Theorem

Let $\boldsymbol{A}$ be a standard polar algebra with countably many points of non-continuity and L the expansion of $\mathrm{QL}_{\boldsymbol{A}}$ by rules $\bar{r} \vdash \bar{0}$, for $r < 1$ and

$$\{(\bar{r}_i \to^{\hat{c}(i)} \varphi_i) \vee \chi \mid r_i <^{\hat{c}(i)} x_i \text{ and } i \leq n\} \vdash_{\mathrm{L}} (\bar{r} \to c(\varphi_1, \dots, \varphi_n)) \vee \chi$$
$$\{(\varphi_i \to^{\hat{c}(i)} \bar{r}_i) \vee \chi \mid x_i <^{\hat{c}(i)} r_i \text{ and } i \leq n\} \vdash_{\mathrm{L}} (c(\varphi_1, \dots, \varphi_n) \to \bar{s}) \vee \chi$$

for each

- *n-ary $c \in \mathcal{L}$ such that $c^{\boldsymbol{A}}$ is non-continuous in $\langle x_1, \dots, x_n \rangle$*
- *and rational r, s such that $r < c^{\boldsymbol{A}}(x_1, \dots, x_n)$ and $s > c^{\boldsymbol{A}}(x_1, \dots, x_n)$*

Then L the least semilinear rational expansion of a polar logic $\mathrm{L}_{\boldsymbol{A}}$ enjoying the Pavelka-style completeness.
Furthermore $\models_{\boldsymbol{A}^{\mathrm{Q}}}$ is its extension by the rule (PIR).

Note: much simpler axiomatizations are know in particular cases, see recent works of Vidal, Bou, Esteva, Godo.

Solving one open problem problem

Theorem

Let $\mathbf{A}$ be the standard product algebra (with $\triangle$). Then the logic $\models_{\mathbf{A}^{\mathrm{Q}}}$ is is the extension of $\mathrm{QL}_{\mathbf{A}}$ by rules $\overline{r} \vdash \overline{0}$, for each $r < 1$ and two infinitary rules:

$$\{(\varphi \to \overline{r}) \vee \chi \mid r > 0\} \vdash (\varphi \to \overline{0}) \vee \chi$$

$$\{(\overline{r} \to \varphi) \vee \chi \mid r < 1\} \vdash \varphi$$

Theorem

Let $\mathbf{A}$ be the standard MV-algebra with $\triangle$. Then the logic $\models_{\mathbf{A}^{\mathrm{Q}}}$ is extension of $\mathrm{QL}_{\mathbf{A}}$ by an infinitary rule:

$$\{(\overline{r} \to \varphi) \vee \chi \mid r < 1\} \vdash \varphi$$

A bonus: simple alternative proofs of known things

Consider e.g. Łukasiewicz logic with additional rule

$$\{\psi \to \varphi^n \mid n > 0\} \vdash \varphi \vee \neg\psi$$

Note that every MV-chain satisfying this rule is simple
and so embeddable into $[0, 1]_Ł$

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And this is trivial because

$$(\psi \to \varphi^n) \vee \chi \vdash_{Ł} \psi \to (\varphi \vee \chi)^n$$