Pavelka-style complete fuzzy logics*

Petr Cintula

Institute of Computer Science, Czech Academy of Sciences, `cintula@cs.cas.cz`

The core agenda of Mathematical Fuzzy Logic is the study of the so-called logics of (left-) continuous t-norms, such as Łukasiewicz logic Ł, Gödel–Dummett logic G, Product logic Π, Hájek logic HL and Esteva–Godo's logic MTL and their expansions by additional connectives. One particular agendum is the study of expansions of these logics by truth constants from $[0, 1]$. Historically, there are two main streams of research in this area:

- The first one follows the work of Jan Pavelka [9], which works with generalized (so-called evaluated) syntax of Łukasiewicz logic, where the basic syntactical object is a pair ⟨formula, real number⟩. This approach was later elaborated mainly by Vilém Novák and his group, see [7, 8]. Due to the evaluated syntax, even other syntactical and semantical notions, notably those of provability and validity, are rendered as *graded* (degree-based) concepts and the completeness theorem is expressed as equality of these two degrees.
- The second approach originated by Petr Hájek [2] who interpreted Pavelka logic in Łukasiweicz logic with truth constants as additional nullary connectives. In this, more traditional, setting the 'natural' notion of completeness just identifies existence of proof with validity in all models, thus the equality of *degrees* of provability and validity is an entirely different form of completeness which Hájek called 'Pavelka-style completeness'.

It is well known that the first approach is tightly connected to Łukasiewicz logics. On the other hand the second approach can be followed for almost all fuzzy logics (see an extensive survey [4, Section 2]); however, the problem of the Pavelka-style completeness was usually disregarded in this stream of research.

The goal of this work is to offer systematic theory of logics with additional rational truth constants with the stress on Pavelka-style completeness (this work is an extension of my upcoming paper [1] and is inspired by previous papers [3, 5, 6]).

Let us fix a propositional language $\mathcal{L}$ expanding that of MTL and its expansion $\mathcal{LR}$ by truth constants (nullary connectives) $\{\overline{r} \mid r \in [0, 1] \cap \mathrm{Q}\}$. An $\mathcal{L}$-algebra

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$\boldsymbol{A}$ is *standard* if its lattice reduct is the real unit interval with the usual order and the set $[0,1] \cap \mathsf{Q}$ is closed under the operations of $\boldsymbol{A}$. By $\boldsymbol{A}^{\mathsf{Q}}$ we denote its expansion into the language $\mathcal{LR}$ by setting $\overline{r}^{\boldsymbol{A}^{\mathsf{Q}}} = r$.

Given a standard algebra $\boldsymbol{A}$, we define the logic $\mathrm{L}_{\boldsymbol{A}}$ as the finitary companion of $\models_{\boldsymbol{A}}$ (i.e., the largest finitary logic contained in the semantical consequence relation given by algebra $\boldsymbol{A}$).

We study three prominent (and in general different) possible expansions of $\mathrm{L}_{\boldsymbol{A}}$: the logics $\models_{\boldsymbol{A}^{\mathsf{Q}}}$ (the logic of the algebra $\boldsymbol{A}$ expanded by rational truth constants interpreted by themselves), $\mathrm{L}_{\boldsymbol{A}^{\mathsf{Q}}}$ (the finitary companion of $\models_{\boldsymbol{A}^{\mathsf{Q}}}$), and $\mathrm{QL}_{\boldsymbol{A}}$ (expansion of $\mathrm{L}_{\boldsymbol{A}}$ by the so-called book-keeping axioms). Furthermore we also explore the class of all *rational expansions* of $\mathrm{L}_{\boldsymbol{A}}$, i.e., all logics laying between $\mathrm{QL}_{\boldsymbol{A}}$ and $\models_{\boldsymbol{A}^{\mathsf{Q}}}$.

We say that a rational expansion L of $\mathrm{L}_{\boldsymbol{A}}$ enjoys the *Pavelka-style completeness*, PSC for short, if for every theory $T \cup \{\varphi\}$ we have:

$$|\varphi|_T^{\mathrm{L}} = ||\varphi||_T, \quad \text{where}$$

$$||\varphi||_T = \inf\{e(\varphi) \mid e \text{ is an } \boldsymbol{A}^{\mathsf{Q}}\text{-model of } T\} \qquad |\varphi|_T^{\mathrm{L}} = \sup\{r \mid T \vdash_{\mathrm{L}} \overline{r} \to \varphi\}$$

The main results (mostly contained already in [1]) can be summarized as:

- The logics $\mathrm{L}_{\boldsymbol{A}^{\mathsf{Q}}}$ and $\mathrm{QL}_{\boldsymbol{A}}$ coincide whenever $\mathrm{L}_{\boldsymbol{A}}$ expands the Łukasiewicz logics. But in general there could be even uncountably many different logics between $\mathrm{L}_{\boldsymbol{A}^{\mathsf{Q}}}$ and $\mathrm{QL}_{\boldsymbol{A}}$.
- The logic $\models_{\boldsymbol{A}^{\mathsf{Q}}}$ enjoys the PSC; is never finitary; and can be axiomatized by adding the infinitary rule $\{\overline{r} \to \varphi \mid r < 1\} \vdash \varphi$ to an arbitrary axiomatization of an arbitrary rational expansion of $\mathrm{L}_{\boldsymbol{A}}$ with PSC.
- If $\boldsymbol{A}$ is polar (i.e. if each connective is mono/antitone in each argument) and all its connectives are continuous, then $\mathrm{QL}_{\boldsymbol{A}}$ enjoys the PSC. Otherwise no finitary rational expansion of $\mathrm{L}_{\boldsymbol{A}}$ enjoys the PSC.
- We have characterized the semilinear rational expansions (i.e., those complete w.r.t. linearly ordered algebras) of any given polar logic $\mathrm{L}_{\boldsymbol{A}}$ enjoying the PSC.
- We have provided an (infinitary) axiomatizations of the logic $\models_{\boldsymbol{A}^{\mathsf{Q}}}$ for $\boldsymbol{A}$ being a polar expansion of $[0,1]_{\text{Ł}}$, $[0,1]_{\text{Ł}_\triangle}$ or $[0,1]_{\Pi_\triangle}$ by continuous connectives.

Besides these results I will present recent progress on the following issues:

- How to axiomatize $\models_{\boldsymbol{A}^{\mathbb{Q}}}$ for a wider class of algebras, in particular for $\boldsymbol{A} = [0,1]_{\Pi}$?
- Is any Pavelka-style complete logic semilinear?
- Would this all work also in the first-order setting?
- Is the restriction to polar algebras really needed?

References

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