Tense Operators on De Morgan posets

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For De Morgan posets, i.e., bounded posets equipped with an antitone involution, the so-called tense operators G and H were already introduced by Chajda and Paseka [3]. They presented also a canonical construction of them using the notion of a time frame. Another approach to tense operators on De Morgan posets has been considered by Figallo and Pelaitay in [5].

Tense operators express the quantifiers "it is always going to be the case that" and "it has always been the case that" and hence enable us to express the dimension of time in our intended logic.

A crucial problem concerning tense operators is their representation. Having a De Morgan posets with tense operators, we can ask if there exists a time frame such that each of these operators can be obtained by our canonical construction. This problem was already solved for the classical propositional calculus by introducing tense operators in Boolean algebras, see [2]. For MV-algebras and for Łukasiewicz-Moisil algebras, the tense operators were introduced by Diaconescu and Georgescu in [4] and the representation problem was solved in [1].

Let $\mathbf{A} = (A; \leq, ', 0, 1)$ be a De Morgan poset. Unary operators G and H on $\mathbf{A}$ are called *tense operators* if they are mappings of A into itself satisfying the following axioms:

(T1) $G(0) = H(0) = 0$, $G(1) = H(1) = 1$,

(T2) $x \leq y$ implies $G(x) \leq G(y)$ and $H(x) \leq H(y)$,

(T3) $x \leq GP(x)$ if $P(x) = H(x')'$ and $x \leq HF(x)$ if $F(x) = G(x')'$.

The structure $\mathbf{A} = (A; \leq, ', G, H, 0, 1)$ is then called a *tense De Morgan poset* and P (or F) is a left adjoint to G (or H, respectively).

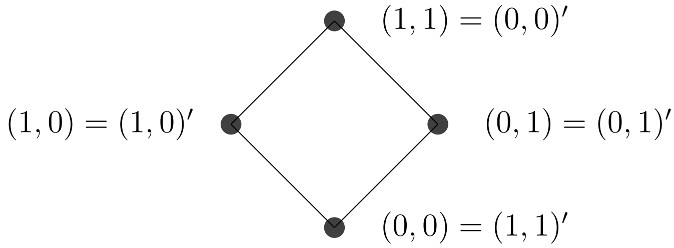


Figure 1: Figure of the underlying poset of the complete De Morgan lattice $\mathbf{M}_2$

A canonical example of a De Morgan poset is the four-element De Morgan poset $\mathbf{M}_2$ depicted in Fig. 1. Recall that $\mathbf{M}_2$, considered as a distributive De Morgan lattice, generates the variety of all distributive De Morgan lattices.

Let T be a non-empty set and R a relation on T such that for all $s \in T$ there are $u, v \in T$ such that $uRsRv$. We then say that (T, R) is a *time frame*. In particular, the cartesian product $\mathbf{M}_2^T$ equipped with the operators $\widehat{G}$ and $\widehat{H}$ such that

$$\widehat{G}(p)(s) = \bigwedge\nolimits_{\mathbf{M}_2}\{p(t) \mid sRt\} \quad \text{and} \quad \widehat{H}(p)(s) = \bigwedge\nolimits_{\mathbf{M}_2}\{p(t) \mid tRs\}$$

for all $p \in M_2^T$ and all $s \in T$ is a tense De Morgan poset.

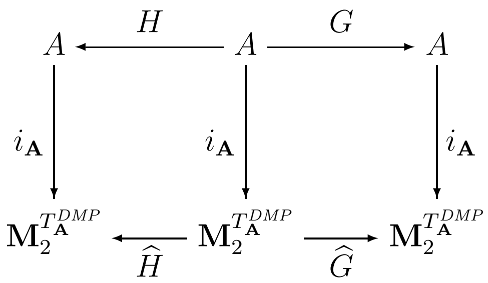
For any De Morgan poset $\mathbf{A} = (A; \leq, ', 0, 1)$, we will denote by $T_{\mathbf{A}}^{\mathrm{DMP}}$ a set of morphisms of De Morgan poset into the four-element De Morgan poset $\mathbf{M}_2$. The elements $\kappa_D \colon A \to M_2$ of $T_{\mathbf{A}}^{\mathrm{DMP}}$ (indexed by proper down-sets D of $\mathbf{A}$ which correspond to morphisms of bounded posets $h_D \colon A \to \{0, 1\}$ such that $h_D(a) = 0$ iff $a \in D$) are morphisms of De Morgan posets defined by the prescription $\kappa_D(a) = (h_D(a), h_D(a')')$ for all $a \in A$.

Now we are able to establish our main results.

Proposition 1. *Let* $\mathbf{A} = (A; \leq, ', 0, 1)$ *be a De Morgan poset. Then the map* $i_{\mathbf{A}} \colon A \to M_2^{T_{\mathbf{A}}^{DMP}}$ *given by* $i_{\mathbf{A}}(a)(s) = s(a)$ *for all* $a \in A$ *and all* $s \in T_{\mathbf{A}}^{DMP}$ *is an embedding of De Morgan posets.*

Theorem 2. *Let* $\mathbf{A} = (A; \leq, ', G, H, 0, 1)$ *be a tense De Morgan poset. Then there is a time frame* $(T_{\mathbf{A}}^{DMP}, R)$ *such that* $\mathbf{A}$ *can be embedded into the tense De*

Morgan poset $(M_2^{T_{\mathbf{A}}^{DMP}}; \leq, ', \widehat{G}, \widehat{H}, 0, 1)$*, i.e., the following diagram commutes:*



Acknowledgments

Both authors acknowledge the support by a bilateral project New Perspectives on Residuated Posets financed by Austrian Science Fund (FWF): project I 1923-N25, and the Czech Science Foundation (GAČR): project 15-34697L.

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