The FEP for residuated lattices via local finiteness of the monoid reducts

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1 Introduction

A class of algebras $\mathcal{K}$ is said to have the *finite embeddability property* (FEP), if for every algebra $\mathbf{A}$ in $\mathcal{K}$ and every *finite* partial subalgebra $\mathbf{B}$ of $\mathbf{A}$, there exists a finite algebra $\mathbf{D}$ in $\mathcal{K}$ such that $\mathbf{B}$ embeds into $\mathbf{D}$. The FEP is a strong property, as it yields decidability for finitely axiomatizable classes and generation by finite algebras for (quasi)varieties.

A residuated lattice is an algebra $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ where $(A, \cdot, 1)$ is a monoid, $(A, \wedge, \vee)$ is a lattice and for all $a, b, c \in A$, we have $ab \leq c$ iff $a \leq c/b$ iff $b \leq a\backslash c$. Since residuated lattices form algebraic semantics for substructural logics, the FEP for a variety of residuated lattices yields the strong finite model property for the corresponding substructural logic.

The FEP was studied for various classes of residuated lattices by W. Blok and C. van Alten in a series of papers. Among other things it is shown that the FEP holds in comutative residuated lattices that satisfy a *knotted inequality* $x^m \leq x^n$ for $m \neq n$, $m \geq 1$, $n \geq 0$. Using results of N. Galatos and P. Jipsen on *residuated frames*, R. Cardona and N. Galatos weaken the condition of commutativity to a balanced equation $(\bar{a})$. Given a vector $\bar{a} = (a_0, a_1, \ldots, a_r)$ of natural numbers such that one of them is zero ($a_i = 0$, for some i) we define the equation:

$$xy_1xy_2 \cdots y_rx = x^{a_0}y_1x^{a_1}y_2 \cdots y_rx^{a_r}. \qquad (\bar{a})$$

If the sum of the a_i's is $r+1$ (there is the same number of x's on the two sides of the equation) then the equation is called *balanced*; for example $xyx = xxy$. It is also shown that further subvarieties of these axiomatized by equations of over the language $\{\vee, \cdot, 1\}$ have the FEP.

Non-balanced identities were not considered, and it is interesting that they imply a knotted identity of the form $x^n = x^m$, for different n, m, obtained by setting all other variables equal to the monoid identity 1. We refer to such an equation as *periodicity*. The methods in our previous work are not obviously applicable to non-balanced equations $(\bar{a})$, and the purpose of this contribution is to show that the FRP holds also for such equations and actually in a stronger way. We also study even weaker versions of these types of equations which we denote by $[\bar{a}]$.

2 Non-balanced equations $(\bar{a})$ and Zimin equations $[\bar{a}]$

An algebra is called *locally finite* if all of its finitelly generated subalgebras are finite. A class of algebras is called *locally finite* if every algebra in it is locally finite. It is easy to see that local finiteness implies the FEP, but such a condition is extremely restrictive for residuated lattices. Instead, requiring local finiteness only for part of the signature turns out to be enough to yield the FEP. In particular, we extent a construction due to McKinsey and Tarski to prove the FEP for suitable varieties.

Lemma 2.1. *Let $\mathcal{V}$ be a variety of residuated lattices axiomatized by equations over the laguage $\{\vee, \cdot, 1\}$. If the monoid reduct of every residuated lattice in $\mathcal{V}$ is locally finite then $\mathcal{V}$ has the FEP.*

Establishing the FEP via local-finiteness of the monoid reduct is very desirable and useful in applications such as in the work of N. Bezhanishvili, L. Spada and N. Galatos (and further recent extensions of the results in their paper), where they establish axiomatizations for certain such varieties of residuated lattices via canonical formulas. This served as part of the motivation to extablish the FEP via local finiteness of the monoid reducts.

Theorem 2.2. *The variety of monoids satisfying a non-balanced $(\bar{a})$ is locally finite. Consequently, any variety of residuated latices axiomatized by any non-balanced $(\bar{a})$ (and possibly by any other additional $\{\vee, \cdot, 1\}$-identities) has the FEP.*

Proof. Using results linking local finiteness of monoids to that of nil semigroups and groups. □

We investigate even more general forms of identities toward that goal. These are defined as substitution instances if $(\bar{a})$ where some of the y_i's are identified.

The family of Zimin words Z_n, for positive integer n, relative to a countably infinite list of variables $x_1, x_2, .,$ is defined inductively by $Z_1 = x_1$ and $Z_{n+1} =$

$Z_n x_{n+1} Z_n$. We usually write x for x_1, y_1 for x_2 and in general y_i for x_{i+1}. So, for example, $Z_4 = xy_1xy_2xy_1xy_3xy_1xy_2xy_1x$.

Given an equation $(\bar{a})$, say $u = v$, we consider a substitution σ which identifies some of the y_i's in such a way that $\sigma(u)$ is an initial subword of some Zimin word Z_n, namely $\sigma(u)w = Z_n$, for some word w. Then we denote the equation $\sigma(u)w = \sigma(v)w$, namely $Z_n = \sigma(v)w$, by $[\bar{a}]$, using angular brackets instead.

For $\bar{a} = (3, 0, 2, 5, 1, 3)$, $(\bar{a})$ is the equation $xy_1xy_2xy_3xy_4xy_5x = x^3y_1y_2x^2y_3x^5y_4xy_5x^3$. In this case the equation $[\bar{a}]$ is $xy_1xy_2xy_1xy_3xy_1xy_2xy_1x = x^3y_1y_2x^2y_1x^5y_3xy_1x^3y_2xy_1x$.

Lemma 2.3. *If a monoid variety satisfies a balanced equation $[\bar{a}]$ and a periodic identity then it is locally finite.*

Proof. Using ideas from symbolic dynamics and personal communication with M. Sapir about a special case. □

We now focus on non-balanced $[\bar{a}]$; these are in the form $Z_n = w$. In the following we set $t = y_{n-1}$, $a = 2^{n-2}$, $b = \sum\{a_i : i < 2^{n-2}\}$ and $c = \sum\{a_i : 2^{n-2} \leq i < 2^{n-1} - 1\}$. Note that Z_n has exactly 2^{n-1} occurrences of x, b sums all the powers of x up to t and c all powers of x after t on the right-hand side; also there are a-many occurrences of x before t and the same amount after t on the left-hand side. Also we set $d = b + c - 2a$ and $g = gcd\{d, |b - a|\}$.

Theorem 2.4. *If $g \neq d$, and x^g appears at least d/g many timed in w, then we have local finiteness. In particular, if $g = 1$ or 2, we have local finiteness.*

For example, using the theorem we can prove the local finiteness for monoids satisfying the equation $Z_4 = x^3yx^2zxytx^4yx^3zx^2yx^3$, as $d = 10$ and $g = 2$.

This is work in progress and time permitting we will present other partial results, such as the local finiteness of all non-balanced equations $[\bar{a}]$ where the left-hand side is Z_3. Also, we will describe methods for proving local finiteness of non-balanced equations $[\bar{a}]$ where x^g does not appear enough times on the right-hand side, or not at all, (so Theorem 2.4 does not apply) such as $Z_4 = x^3yx^3zx^3ytx^3yx^3zx^2yx$, for which $d = 10$ and $g = 5$.