A new invariant for projective MV-algebras

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Introduction

The ultimate goal

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Is there a decision procedure to determine if they are equivalent up to a translation?

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Is there a decision procedure to determine if they are equivalent up to a translation?

That is, is there a computable procedure to determine if there exits two substitutions $s\colon \mathbf{Fm}(X) \to \mathbf{Fm}(Y)$ and $t\colon \mathbf{Fm}(Y) \to \mathbf{Fm}(X)$, such that

$$\alpha \vdash_{Ł_\infty} \gamma \iff \beta \vdash_{Ł_\infty} s(\gamma)$$

and

$$\beta \vdash_{Ł_\infty} \delta \iff \alpha \vdash_{Ł_\infty} t(\delta) \ ?$$

Introduction

Algebraic perspective

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Algebraic perspective

An **MV-algebra** is an algebraic structure $(M, \oplus, \neg, 0)$ where:

- $(M, \oplus, 0)$ is a commutative monoid,
- $\neg$ is a unary operation,
- $\neg$ and $\oplus$ satisfy the following
  - $\neg\neg x = x$,
  - $x \oplus \neg 0 = \neg 0$, and
  - $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

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Standard example: $[0,1]_{MV} = ([0,1], \oplus, \neg, 0)$ where

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Chang's completeness theorem states that $[0,1]_{MV}$ generates the variety of MV-algebras.

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Can we find a computable complete invariant for finitely presented MV-algebras?

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Geometric persperctive

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Geometric persperctive

A **rational polyhedron** $P \subseteq [0,1]^n$ is the pointset union of finitely many convex set with rational vertices.

That is, there exists a finite set $K \subseteq [0,1]^n \cap \mathbb{Q}^n$ and a $L \subseteq \mathcal{P}(K)$

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Geometric perspective

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That is, there exists a finite set $K \subseteq [0,1]^n \cap \mathbb{Q}^n$ and a $L \subseteq \mathcal{P}(K)$ such that

$$P = \bigcup \{\mathrm{conv}(S) \mid S \in L\}.$$

Definition

A map $\eta\colon P \subseteq [0,1]^n \to Q \subseteq [0,1]^m$ is called a **$\mathbb{Z}$-map** if it satisfies the following conditions:

(i) η is continuous,

(ii) piecewise (affine) linear, and each linear piece of η has integer coefficients.

Theorem (Marra, Spada – 2012)

The category of finitely presented MV-algebras is dually equivalent to the category of rational polyhedra with $\mathbb{Z}$-maps.

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Geometric persperctive

Theorem (Marra, Spada – 2012)

The category of finitely presented MV-algebras is dually equivalent to the category of rational polyhedra with $\mathbb{Z}$*-maps.*

Can we find a computable complete invariant for rational polyhedra (under $\mathbb{Z}$-homeomorphisms)?

Introduction

Geometric persperctive: Invariants

Denominator:

For v in $\mathbb{Q}^n$ we let **den**(v) denote the least common denominator of the coordinates of v.

Let $P \subseteq [0,1]^n$ be a rational polyhedron and $d = 1, 2, \ldots$ be an integer number. The number $\#_d$ of rational points of denominator d in P is an invariant (under $\mathbb{Z}$-homeomorphisms).

Introduction

Geometric persperctive: Invariants

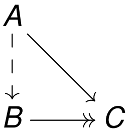
Rational Measure(s):

[Mundici, D., The Haar theorem for lattice-ordered abelian groups with order-unit. Discrete and Continuous Dynamical Systems, 21 (2008) 537-549]

Projective Algebras

Projective Algebras

Let $\mathcal{K}$ be a class of algebras. An algebra $A \in \mathcal{K}$ is **projective** if:



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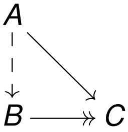
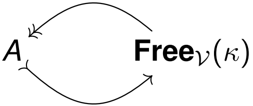


If $\mathcal{V}$ is a variety (equational class) of algebras then $A \in \mathcal{V}$ is projective iff there exists a cardinal κ such that



Projective Algebras
Number of retractions

Definition

Let A be the image of an idempotent endomorphism of

$$\mathcal{M}([0,1]^n) = \{f\colon [0,1]^n \to [0,1] \mid f \text{ is a McNaughton map}\}.$$

The **multiplicity (r(A))** of A is the number of distinct idempotent endomorphisms of $\mathcal{M}([0,1]^n)$ whose range is A if this number is finite, and ∞ otherwise.

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Can we effectively compute $\mathrm{r}(A)$*?*

Theorem

Let A be an MV-algebra. Then TFAE:

(a) *A is finitely generated and projective;*

(b) *there exist* $n \in \{1, 2, \ldots\}$ *and a rational polyhedron* $P \subseteq [0, 1]^n$ *such that*

  (i) *P is contractible,*

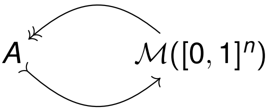
  (ii) $P \cap \{0, 1\}^n \neq \emptyset$,

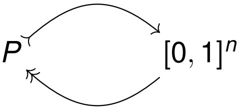
  (iii) *for each* $v \in P \cap \mathbb{Q}^n$ *there exist* $w \in \mathbb{Z}^n$ *and* $\varepsilon > 0$ *such that the convex segment* $\mathrm{conv}(v, v + \varepsilon(w - v))$ *is contained in P (⇔ P is strongly regular ⇔ P is anchored), and*

  (iv) $A \cong \{f \restriction_P \mid f : [0, 1]^n \to [0, 1] \text{ is a McNaughton map}\} = \mathcal{M}(P)$

Counting Retractions

Dual translation





Counting Retractions

Dual translation

Lemma

Let $\rho\colon \mathcal{M}([0,1]^n) \to \mathcal{M}([0,1]^n)$*, be a retraction onto* A *(i.e., an idempotent homomorphism such that* $\rho(\mathcal{M}([0,1]^n)) = A$*). The* $\mathbb{Z}$*-map* $\sigma = (\rho(x_1), \ldots, \rho(x_n))$ *is the dual of* ρ*. Then the map* $\tau \mapsto \tau([0,1]^n)$ *yields a one-one correspondence between*

(a) $\mathbb{Z}$*-retractions such that* $\mathcal{M}(\tau)(\mathcal{M}([0,1]^n) = A$*;*

(b) *rational polyhedra* $Q \subseteq [0,1]^n$ *such that* $\sigma \restriction Q\colon Q \cong_{\mathbb{Z}} \sigma([0,1]^n)$*.*

Counting Retractions

Close Domains: Finite vs Infinite

Counting Retractions

Close Domains: Finite vs Infinite

Theorem

Let $A \subseteq \mathcal{M}([0,1]^n)$ *and* $\rho\colon \mathcal{M}([0,1]^n) \to \mathcal{M}([0,1]^n)$ *be a retraction onto* A*.*

Let $\sigma = (\rho(x_1), \ldots, \rho(x_n))\colon [0,1]^n \to [0,1]^n$ *be the dual* $\mathbb{Z}$*-map of* ρ*. Then TFAE:*

(a) $\sigma([0,1]^n)$ *is a closed domain in* $[0,1]^n$*, that is, it coincides with the closure of its interior in* $[0,1]^n$*;*

(b) *The number* $\mathrm{r}(A)$ *of retractions of* $\mathcal{M}([0,1]^n)$ *onto* A *is finite.*

Counting Retractions
Close Domains: Finite vs Infinite

Key step of the proof:

Lemma

Let $\eta\colon [0,1]^n \to [0,1]^n$ be a $\mathbb{Z}$-map and $P, Q \subseteq [0,1]^n$ be rational polyhedra satisfying the following conditions:

(i) *Both $\mathrm{int}(P)$ and $\mathrm{int}(Q)$ are connected,*

(ii) *$P = \mathrm{cl}(\mathrm{int}(P))$ and $Q = \mathrm{cl}(\mathrm{int}(Q))$ (closed domains),*

(iii) *$\eta(P) = \eta(Q)$,*

(iv) *$\eta{\restriction}P\colon P \cong_{\mathbb{Z}} \eta(P)$ and $\eta{\restriction}Q\colon Q \cong_{\mathbb{Z}} \eta(Q)$.*

Then either $P = Q$ or $\mathrm{int}(P) \cap \mathrm{int}(Q) = \emptyset$.

Counting Retractions

Close Domains

Theorem

Let $A \subseteq \mathcal{M}([0,1]^n)$ and $\rho\colon \mathcal{M}([0,1]^n) \to \mathcal{M}([0,1]^n)$ be a retraction onto A. Let $\sigma = (\rho(x_1), \ldots, \rho(x_n))$ be the dual map of ρ and $P = \sigma([0,1]^n)$.
If P is a closed domain, the multiplicity $\mathrm{r}(A)$ is computable from σ.

Invariant

Issues with r(A):

- It is not invariant under isomorphisms.
- Depends on $\mathcal{M}([0,1]^n)$.
- There is always an n and a $B \cong A$, such that $B \subseteq \mathcal{M}([0,1]^n)$ and $r(B) = \infty$.

Invariant

Definition

Let B be a finitely generated projective MV-algebra and k the smallest number of generators of B. Then the *index* $\mathrm{i}(B) \in \{1, 2, \ldots\} \cup \{\infty\}$ of B is the supremum of the multiplicities of the images A of idempotent endomorphisms of $\mathcal{M}([0,1]^k)$, letting A range over all isomorphic copies of B.

Invariant

Finite - Infinite

Theorem

Let $A \subseteq \mathcal{M}([0,1]^n)$ and $\rho\colon \mathcal{M}([0,1]^n) \to \mathcal{M}([0,1]^n)$ be a retraction onto A.
Let $\sigma = (\rho(x_1), \ldots, \rho(x_n))$ be the dual $\mathbb{Z}$-map of ρ. Then TFAE:

(a) $\sigma([0,1]^n)$ *is a closed domain in* $[0,1]^n$*;*

(b) *The index* $\mathrm{i}(A)$ *is finite.*

Open Problems

- Effective calculation of i.
- Compare i with the rational measure.
- Are the $\#_d$ collectively complete?
- Isomorphism problem of fintely presented algebras.

Thank you for your attention!

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