A new invariant for projective MV-algebras

Leonardo Manuel Cabrer
(Based on a joint work with D. Mundici)

Institute of Computer Languages, Technische Universität Wien, Favoritenstrasse 9-11, A-1040, Wien, Austria, leonardo.cabrer@logic.at

Markov's unrecognizability theorem is to the effect that the isomorphism problem for finitely presented ℓ-groups is undecidable [9, 6]. However, the isomorphism problem for finitely presented unital ℓ-groups (and MV-algebras) is still open [7, §20.3, Problem 1].

An important difference between the isomorphism problems of these structures is that the duality between finitely presented MV-algebras and rational polyhedra is a rich source of computable invariants, such as: the number of rational points of a given denominator $d = 1, 2, \ldots$ in the maximal spectral space μ_A, [7, Proposition 3.15 and Theorem 4.16]; the rational measure of the t-dimensional part of μ_A, $(t = 0, \ldots, \dim(\mu_A))$, [7, §14], [8].

In this work we concentrate our efforts on the study of finitely presented projective MV-algebras. From the Baker-Beynon duality [1, 2, 3] one easily obtains that finitely generated projective ℓ-groups coincide with the finitely presented ones. This is not the case for MV-algebras, where finitely generated projective MV-algebras are a proper subclass of finitely presented MV-algebras. A geometric characterisation of the rational polyhedra dual to finitely generated projective MV-algebras was presented in [4, 5].

We introduce a new numerical invariant for projective MV-algebras and study its properties. First we need the following notation: the *multiplicity* $\mathsf{r}(A)$ of a retract A of the n-generator free MV-algebra ($\mathcal{M}([0,1]^n)$ the MV-algebra of McNaughton functions from $[0,1]^n$ to $[0,1]$) is the number of distinct retractions of $\mathcal{M}([0,1]^n)$ onto A if this number is finite, and ∞ otherwise. The *index* $\iota(B) \in \{1, 2, \ldots\} \cup \{\infty\}$ of a finitely generated projective MV-algebra B is the supremum of the multiplicities of all $A \cong B$ such that $A \subseteq \mathcal{M}([0,1]^\kappa)$, with k the smallest number of generators of B.

It is easy to see that the index of every finitely generated free MV-algebra is 1. Also, the index of one-generator projective MV-algebras is simple to determine. Indeed if $B \not\cong \{0,1\}$ is a one-generator projective MV-algebra. Then $\iota(B) = 2$, unless the maximal spectral space μ_B contains an element $\mathfrak{m}$ such that $B/\mathfrak{m} \cong \{0, 1/2, 1\}$ and $\mu_B \setminus \{\mathfrak{m}\}$ is disconnected in which case $\iota(B) = 1$. This is discreet behaivoir of the index only holds for one-generator projective MV-algebras. For each m there exist 2-generated projective MV-algebras A such that $\iota(A) \geq m$.

Our main results are: (1) to determine when the index and the multiplicity of retract A of $\mathcal{M}([0,1]^n)$ are finite; (2) the computability of the multiplicity when we are given a retraction from $\mathcal{M}([0,1]^n)$ onto A.

Theorem 1. *Suppose A is a retract of $\mathcal{M}([0,1]^n)$. Then the following are equivalent:*

(a) μ_A is a closed domain in $[0,1]^n$, that is, it coincides with the closure of its interior in $[0,1]^n$;

(b) $\mathsf{r}(A)$ is finite.

Corollary 2. *Let k be the smallest number of generators of a finitely generated projective MV-algebra B. Then the index of B is finite iff the maximal spectral space of B is homeomorphic to a regular domain in $[0,1]^k$.*

From the geometric characterisation of projective MV-algebras presented in [4], we can prove that given a finite presentation of an MV-algebra A, checking whether A is a projective MV-algebra is not a decidable problem. Therefore, in the following result, to describe a finitely presented MV-algebra we provide an idempotent endomorphism of the free algebra $\mathcal{M}([0,1]^n)$ instead of a presentation. (Observe that an endomorphism can be presented by $f(\pi_1), \ldots, f(\pi_n)$, the images of $\pi_1, \ldots, \pi_n$ the free generators of $\mathcal{M}([0,1]^n)$. Moreover, given $f(\pi_1), \ldots, f(\pi_n)$ it is decidable of f is idempotent.)

Theorem 3. *Let $f\colon \mathcal{M}([0,1]^n) \to \mathcal{M}([0,1]^n)$ be an idempotent endomorphism and $A = f(\mathcal{M}([0,1]^n))$. The multiplicity $\mathsf{r}(A)$ is computable from generators $f(\pi_1), \ldots, f(\pi_n)$ of A.*

Wether the index is computable or not remains an open problem.

References

[1] K. A. Baker, Free vector lattices, Canad. J. Math., 20 (1968) 58–66.

[2] W. M. Beynon, On rational subdivisions of polyhedra with rational vertices, Canad. J. Math., 29 (1977) 238–242.

[3] W. M. Beynon, Applications of duality in the theory of finitely generated lattice-ordered abelian groups, Canad. J. Math., 29 (1977) 243–254.

[4] L. Cabrer, Rational simplicial geometry and projective lattice-ordered abelian groups, arXiv:1405.7118v1 [math.RA] 28 May 2014

[5] L. Cabrer, D.Mundici, Rational polyhedra and projective lattice-ordered abelian groups with order unit, Commun. Contemp.Math., 14.3 (2012) 1250017 DOI: 10.1142/S0219199712500174.

[6] A. M. W. Glass, J. J. Madden, The word problem versus the isomorphism problem, J. Lond. Math. Soc., 30 (1984) 53–61.

[7] D. Mundici, Advanced Łukasiewicz calculus and MV-algebras, Trends in Logic, vol. 35, Springer, Berlin, 2011.

[8] D. Mundici, Invariant measure under the affine group over **Z**, Combin. Probab. Comput., 23 (2014) 248–268.

[9] M. A. Shtan'ko, Markov's theorem and algorithmically non-recognizable combinatorial manifolds, Izv. Math., 68 (2004) 207–224.