# $\omega$-Semigroups and the Fine Classification of Borel Subsets of Finite Ranks of the Cantor Space

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**Abstract**

The algebraic study of formal languages draws the equivalence between $\omega$-regular languages and subsets of finite $\omega$-semigroups. The $\omega$-regular languages being the ones characterised by second order monadic formulas. Within this framework, in [1]we provide a characterisation of the algebraic counterpart of the Wagner hierarchy: a celebrated hierarchy of $\omega$-regular languages. For this, we adopt a hierarchical game approach and translate, from the $\omega$-regular language to the $\omega$-semigroup context, the very few elements of the Wadge theory that the Wagner hierarchy is concerned with. We also define a reduction relation on subsets of finite $\omega$-semigroups by means of a Wadge-like infinite two-player game, and then give a description of the resulting hierarchy of such subsets. Finally, we prove that this algebraic hierarchy is isomorphic to the Wagner hierarchy.

We propose to extend this algebraic approach from the Wagner hierarchy to the first levels of the Borel hierarchy (of subsets of the Cantor Space) by considering infinite $\omega$-semigroups – obtained as combinations of finite $\omega$-semigroups – equipped with pseudoinverses.

## 1 Introduction

The notion of an $\omega$-semigroup was first introduced by Jean-Eric Pin as a generalisation of semigroups [6, 8]. In the case of finite structures, these objects represent a convincing algebraic counterpart to automata reading infinite words: given any finite Büchi automaton, one can build a finite $\omega$-semigroup that recognises – in an algebraic sense – the same language, and conversely, given any finite $\omega$-semigroup recognising a certain language, one can build a finite Büchi automaton recognising the same language.

**Definition 1.** *(see [7], p. 92). An $\omega$-semigroup is an algebra consisting of two components, $S = (S_+, S_\omega)$, and equipped with the following operations:*

- *a binary operation on $S_+$, denoted multiplicatively, such that $S_+$ equipped with this operation is a semigroup;*
- *a mapping $S_+ \times S_\omega \to S_\omega$, called mixed product, which associates with each pair $(s,t) \in S_+ \times S_\omega$ an element of $S_\omega$ , denoted by $st$, and such that for every $s,t \in S_+$ and for every $u \in S_\omega$, then $s(tu) = (st)u$;*
- *a surjective mapping $\pi_S : {S_+}^\omega \to S_\omega$, called infinite product, which is compatible with the binary operation on $S_+$ and the mixed product in the following sense: for every strictly increasing sequence of integers $(k_n)_{n>0}$, for every sequence $(s_n)_{n\geq 0} \in {S_+}^\omega$, and for every $s \in S_+$, then*

$$\pi_S(s_0 s_1 \dots s_{k_1-1}, s_{k_1} \dots s_{k_2-1}, \dots) = \pi_S(s_0, s_1, s_2, \dots),$$

$$s\pi_S(s_0, s_1, s_2, \dots) = \pi_S(s, s_0, s_1, s_2, \dots).$$

Intuitively, an $\omega$-semigroup is simply a semigroup equipped with a suitable infinite product.

**Example 2.** Let A be any alphabet. The $\omega$-semigroup $A^\infty = (A^+, A^\omega)$ equipped with the usual concatenation is the free $\omega$-semigroup over the alphabet A.

A congruence of an $\omega$-semigroup $S = (S_+, S_\omega)$ [7] is a pair $(\sim_+, \sim_\omega)$, where $\sim_+$ is a semigroup congruence on $S_+$, $\sim_\omega$ is an equivalence relation on $S_\omega$, and these relations are stable for the infinite and the mixed products.

A slightly extended version of the notion of $\omega$-semigroup is the one of pointed $\omega$-semigroup, adapted from the notion of pointed semigroup introduced by Sakarovitch in [9]. A *pointed* $\omega$-semigroup is a pair $(S, X)$, where $S$ is an $\omega$-semigroup and $X$ is a subset of $S$. A mapping $\phi : (S, X) \to (T, Y)$ is a morphism of pointed $\omega$-semigroups if $\phi : S \to T$ is a morphism of $\omega$-semigroups such that $\phi^{-1}(Y) = X$. All notions of $\omega$-subsemigroups, quotient, and division are easily adapted in the context of pointed $\omega$-semigroups.

**Definition 3.** *Let $S$ and $T$ be two $\omega$-semigroups. A surjective morphism of $\omega$-semigroups $\phi : S \to T$* recognises *a subset $X$ of $S$ if there exists a subset $Y$ of $T$ such that $\phi^{-1}(Y) = X$. By extension, one also says in this case that the $\omega$-semigroup $T$ recognises $X$. In addition, a congruence $\sim$ on $S$ recognises the subset $X$ of $S$ if the natural morphism $\pi : S \to S/\sim$ recognises $X$.*

Wilke was the first to give the appropriate algebraic counterpart to finite automata reading infinite words [13]. In addition, he established that the $\omega$-languages recognised by finite $\omega$-semigroups are exactly the ones recognised by Büchi automata.

**Theorem 4** (Wilke)**.** *An $\omega$-language is recognised by a finite $\omega$-semigroup if and only if it is $\omega$-regular.*

The syntactic pointed $\omega$-semigroup of an $\omega$-regular language is – up to isomorphism – the unique minimal (for the division) pointed $\omega$-semigroup recognising this language. Moreover, it turns out that this algebraic notion coincides with the one of Wadge degree that comes from descriptive set theory: a Wadge degree is an equivalence class of subsets of the Cantor space under the Wadge preordering which is itself defined as $A \leq_W B$ if and only if there exists some continuous function $f$ on the Cantor space such that $A = f^{-1}B$. Indeed, the Wadge degree is a syntactic invariant in the following sense: if two $\omega$-regular languages have the same syntactic image, then they also have the same Wadge degree. Therefore, the Wadge degree of every $\omega$-regular language can be characterised by some algebraic invariant on its syntactic image.

We define a reduction relation on pointed $\omega$-semigroups by means of an infinite two-player game. This reduction induces a hierarchy of Borel $\omega$-subsets, called the $\mathbb{SG}$-hierarchy. Many results of the Wadge theory [12] also apply in this framework and provide a detailed description of the $\mathbb{SG}$-hierarchy.

**Definition 5.** *Let $S = (S_+, S_\omega)$ and $T = (T_+, T_\omega)$ be two $\omega$-semigroups, and let $X \subseteq S_\omega$ and $Y \subseteq T_\omega$ be two $\omega$-subsets.*
*The game $\mathbb{SG}\big((S, X), (T, Y)\big)$ is an infinite two-player game with perfect information, where Player I is in charge of $X$, Player II is in charge of $Y$, and players I and II take turn playing elements of $S_+$ and $T_+$, respectively. Player II begins. Unlike Player I, Player II is allowed to skip her turn provided she plays infinitely many moves. After $\omega$ turns each, players I and II have produced respectively two infinite sequences $(s_0, s_1, \ldots) \in S_+^\omega$ and $(t_0, t_1, \ldots) \in T_+^\omega$.*

$$\text{Player II wins } \mathbb{SG}\big((S, X), (T, Y)\big) \text{ if and only if}$$
$$\pi_S(s0, s1, \ldots) \in X \iff \pi_T(t_0, t_1, \ldots) \in Y.$$

The $\mathbb{SG}$-reduction is defined by $X \leq_{\mathbb{SG}} Y$ if and only if Player *II* has a winning strategy in the underlying game $\mathbb{SG}\big((S, X), (T, Y)\big)$.

We study this algebraic hierarchy on a sub-family of infinite $\omega$-semigroups obtained as combinations of finite $\omega$-semigroups and show, by paying attention to the pseudoinverses involved, that this $\mathbb{SG}$-hierarchy corresponds to the algebraic counterpart of the Wadge hierarchy of Borel subsets of finite ranks of the Cantor space: both hierarchies are isomorphic and $\omega$-languages of a given Wadge degree are recognised by $\omega$-semigroups located as the same level in the $\mathbb{SG}$-hierarchy.

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