PWK logic and Involutive Bisemilattices
Part I

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Joint work with J. Gil-Férez, F. Paoli and L. Peruzzi

Outline

1. Introduction to the Kleene family

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2. Paraconsistent Weak Kleene logic
3. Involutive bisemilattices

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4. Part II: AAL approach to Paraconsistent Weak Kleene logic

The Kleene tables

Strong Kleene tables:

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
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$\neg$	
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Weak Kleene tables:

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0	0	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
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$\vee$	0	$\frac{1}{2}$	1
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Kleene's family: strong logics

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- Strong Kleene logic: $\langle \mathbf{SK}, \{1\} \rangle$
- The Logic of Paradox, LP: $\langle \mathbf{SK}, \{1, \frac{1}{2}\} \rangle$

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- Paraconsistent Weak Kleene logic, PWK: $\langle \mathbf{WK}, \{1, \frac{1}{2}\} \rangle$

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Our aim: investigate PWK, the "ugly duckling" in the family of Kleene's logics

Paraconsistent Week Kleene, the definition

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- The matrix: **PWK** $= \langle \mathbf{WK}, \{1, \frac{1}{2}\}\rangle$

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Basic facts

- $\alpha, \alpha \to \beta \nvDash_{\textsf{PWK}} \beta$,
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- Classical contradictions are satisfiable (ex: $\alpha \wedge \neg\alpha$)
- $\alpha \wedge \neg\alpha \nvDash_{\mathsf{PWK}} \beta$
- Deduction Theorem does not hold

A Hilbert calculus for PWK

Definition (HPWK)

A1. $\alpha \to (\beta \to \alpha)$;

A2. $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$;

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[RMP] $\dfrac{\alpha \qquad \alpha \to \beta}{\beta}$ provided that $\mathrm{var}(\alpha) \subseteq \mathrm{var}(\beta)$.

Completeness Theorem

Proposition

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Theorem (Completeness)

$$\mathrm{HPWK} = \mathrm{PWK}$$

A closer look to WK

$$\mathbf{WK} = \langle \{0, 1, \tfrac{1}{2}\}, \vee, \wedge, \neg, 0, 1 \rangle$$

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$\frac{1}{2}$
|
1
|
0

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$$\begin{array}{c} \frac{1}{2} \\ | \\ 1 \\ | \\ 0 \end{array} \qquad\qquad \begin{array}{c} 1 \\ | \\ 0 \\ | \\ \frac{1}{2} \end{array}$$

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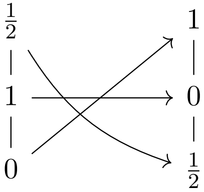

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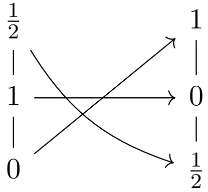

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Counterexample to absorption:

$$1 \wedge (1 \vee \frac{1}{2}) = \frac{1}{2} \neq 1$$

Involutive bisemilattices

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The class of involutive bisemilattices is a variety denoted by $\mathcal{IBSL}$.

Examples

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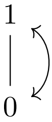
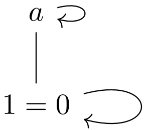
Also, the 2-element semilattice with zero, endowed with identity as its unary fundamental operation, is an involutive bisemilattice.

$\mathbf{B}_2 =$ 

$\mathbf{S}_2 =$ 

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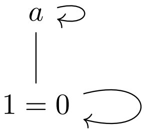
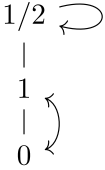
$\mathbf{S}_2 =$ 

$\mathbf{WK} =$ 

About $\mathcal{IBSL}$

Proposition

If $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ *is an involutive bisemilattice, then* $\langle B, \wedge, \vee\rangle$ *is a distributive bisemilattice, that is, the equation*

$$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$$

and its dual are satisfied.

WK and $\mathcal{IBSL}$

Theorem 1

The only nontrivial subdirectly irreducible bisemilattices are $\mathbf{WK}$, $\mathbf{S}_2$, and $\mathbf{B}_2$, up to isomorphism.

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$\mathcal{IBSL}$ is the variety generated by $\mathbf{WK}$.

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Corollary 1

$\mathcal{IBSL}$ is the variety generated by $\mathbf{WK}$.

Corollary 2

The only nontrivial proper subvarieties of $\mathcal{IBSL}$ are the disjoint varieties $\mathcal{BA}$ of Boolean algebras and $\mathcal{SL}$ of semilattices with zero.

The structure of involutive bisemilattices

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Proposition 2

Let $\mathbf{B}$ be an involutive bisemilattice. For every positive $c \in B$, the segment $[\neg c, c]$ is the universe of a Boolean algebra $\mathbf{C}$ under the restrictions of the non-nullary operations of $\mathbf{B}$, and with $0^{\mathbf{C}} = \neg c$, $1^{\mathbf{C}} = c$.

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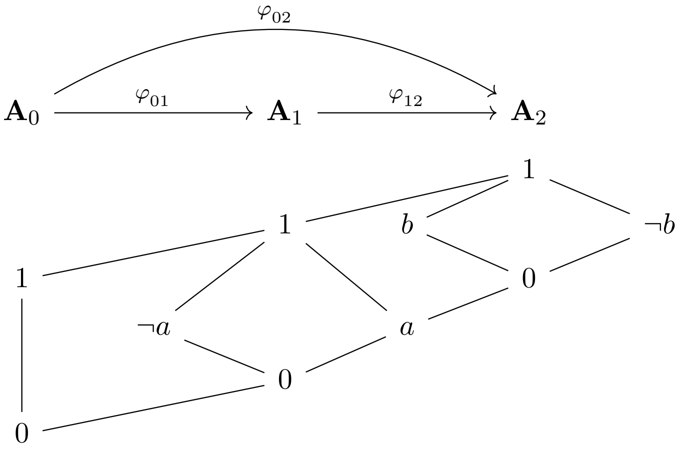
- if $g \in \nu$ is a constant, then $g^{\mathbf{T}} = g^{\mathbf{A}_{i_0}}$.

Płonka sums: example



Płonka sums representation

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2. If $\mathbf{B}$ is an involutive bisemilattice, then $\mathbf{B}$ is isomorphic to the Płonka sum over the direct system $\left(\{\mathbf{A}_c\}_{c\in P(\mathbf{B})}, \geqslant, \{\varphi_{cd} : c, d \in P(\mathbf{B}) \text{ and } \mathbf{A}_c \leqslant \mathbf{A}_d\}\right)$, where each $\mathbf{A}_c = [\neg c, c]$ is a Boolean algebra and $\varphi_{cd} : [\neg c, c] \to [\neg d, d]$ is given by $\varphi_{cd}(a) = \neg d \vee a$.

Płonka sums representation

Theorem

1. If $\mathtt{T}$ is a direct system of Boolean algebras, then the Płonka sum $\mathbf{T}$ over $\mathtt{T}$ is an involutive bisemilattice.
2. If $\mathbf{B}$ is an involutive bisemilattice, then $\mathbf{B}$ is isomorphic to the Płonka sum over the direct system $\left(\{\mathbf{A}_c\}_{c\in P(\mathbf{B})}, \geqslant, \{\varphi_{cd} : c, d \in P(\mathbf{B})$ and $\mathbf{A}_c \leqslant \mathbf{A}_d\}\right)$, where each $\mathbf{A}_c = [\neg c, c]$ is a Boolean algebra and $\varphi_{cd} : [\neg c, c] \to [\neg d, d]$ is given by $\varphi_{cd}(a) = \neg d \vee a$.

Corollary

$\mathcal{IBSL}$ is the variety satisfying exactly the regular identities satisfied by $\mathcal{BA}$.

The end (of Part I)

Thank you for your attention!!