On Paraconsistent Weak Kleene Logic and Involutive Bisemilattices - Part II

Stefano Bonzio[1], José Gil-Férez[2], Francesco Paoli[1], and Luisa Peruzzi[1]

[1] University of Cagliari, Italy
[2] Vanderbilt University, Nashville, TN
jose.gil-ferez@vanderbilt.edu

Extended abstract

Paraconsistent Weak Kleene logic, PWK, was essentially introduced by Halldén [8] and, in a completely independent way, by Prior [12], and is the logic based in the weak Kleene tables (see [9, § 64]) with 1 and $\frac{1}{2}$ as designated values. That is, it is the logic $\vDash_{\mathsf{PWK}}$ defined semantically by the matrix $\mathsf{PWK} = \langle \mathbf{WK}, \{\frac{1}{2}, 1\}\rangle$, where $\mathbf{WK} = \langle \{0, 1, \frac{1}{2}\}, \wedge, \vee, \neg, 0, 1\rangle$ is the algebra given by the tables:

$\wedge$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	1	$\frac{1}{2}$	1

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

The aim of this contribution is the study of PWK under the light of Abstract Algebraic Logic. PWK, unlike the other three logics in the Kleene family (Strong Kleene logic [9], the Logic of Paradox [11, 13, 14], and Bochvar's logic [2]), is often passed over in silence in the main reviews on finite-valued logics. Most of the extant literature concerns the philosophical interpretation of the third value [1, 3, 6, 8, 15] and a discussion of the so-called *contamination principle* (any sentence containing a subsentence evaluated at $\frac{1}{2}$ is itself evaluated at $\frac{1}{2}$), as well as proof systems of various kinds [1, 4, 5, 7].

In the first part, we gave a Hilbert system for PWK, introduced some algebraic structures called *involutive bisemilattices*, which form a variety, $\mathcal{IBSL}$, and proved that $\mathcal{IBSL}$ is generated as a variety by the sole algebra $\mathbf{WK}$. Moreover, we described an algebraic construction called Płoka sums (see [10]), and proved that involutive bisemilattices can be regarded as the Płonka sums whose fibres are Boolean algebras.

In this second part, we study PWK by recourse to the toolbox of Abstract Algebraic Logic. It is not inappropriate to wonder whether the variety $\mathcal{IBSL}$ is the actual algebraic counterpart of the logic PWK. Such a guess stands to reason, for PWK is the logic defined by the matrix **PWK** with $\mathbf{WK}$ as an underlying algebra, and $\mathcal{IBSL}$ is the variety generated by $\mathbf{WK}$. We show though that $\mathcal{IBSL}$ is not the equivalent algebraic semantics of any algebraisable logic, and furthermore, PWK is not algebraisable, since it is not even protoalgebraic. We also show that PWK is not selfextensional either.

We start by characterising the Leibniz congruence of the models of PWK, what allows us to prove that the class $\mathsf{Alg}^*(\mathrm{PWK})$ of the algebraic reducts of the reduced models of PWK is a subclass of $\mathcal{IBSL}$. As a consequence we obtain the following:

Theorem A. *The intrinsic variety of* PWK *is* $\mathbb{V}(\mathsf{Alg}^*(\mathrm{PWK})) = \mathcal{IBSL}$.

Next, we fully characterise the deductive PWK-filters on members of $\mathcal{IBSL}$ and the reduced matrix models of PWK. More in detail, given an involutive bisemilattice $\mathbf{B}$, we define the set of *positive* elements of $\mathbf{B}$ as the set $P(\mathbf{B}) = \{c \in B : 1 \leqslant c\}$, that is, those elements that are above of 1 in the order given by $\vee$, and we prove the following:

Theorem B. *An algebra* $\mathbf{B} \in \mathsf{Alg}^*(\mathrm{PWK})$ *if and only if* $\mathbf{B}$ *is an involutive bisemilattice and for every* $a < b$ *positive elements, there is* $c \in B$ *such that*

$$1 \leqslant \neg b \vee c \quad \textit{but} \quad 1 \nleqslant \neg a \vee c.$$

Moreover, $\langle \mathbf{B}, F\rangle \in \mathsf{Mod}^*(\mathrm{PWK})$ *if and only if* $\mathbf{B}$ *is an involutive bisemilattice satisfying the above condition and* $F = P(\mathbf{B})$.

As a consequence, we obtain that PWK is truth-equational, since for every involutive bisemilattice $\mathbf{B}$, $P(\mathbf{B})$ can be equationally described by the equation $1 \vee x \approx x$. This result finishes the exact location of PWK within the Leibniz hierarchy. Interestingly enough, we show that the class $\mathsf{Alg}^*(\mathrm{PWK})$ is not even a generalised quasivariety, since it is not closed under quotients nor subalgebras.

Next, we investigate PWK with the methods of second-order AAL. We prove that the classes $\mathsf{Alg}^*(\mathrm{PWK})$ and $\mathsf{Alg}(\mathrm{PWK})$, are different. Furthermore, the class $\mathsf{Alg}(\mathrm{PWK})$ is formed by the involutive bisemilattices with at most one *fix* element (i.e., $\neg c = c$), which can be expressed by a quasiequation. Hence, we obtain the following:

Theorem C. $\mathsf{Alg}(\mathrm{PWK})$ *is the quasivariety of involutive bisemilattices satisfying the quasiequation*

$$\neg x \approx x \;\&\; \neg y \approx y \Rightarrow x \approx y.$$

We remark the fact that PWK is one of the few natural examples of a logic whose algebraic counterpart, i.e. $\mathsf{Alg}(\mathrm{PWK})$, is a quasivariety but not a variety. Finally, using the representation of an involutive bisemilattice as a Płonka sum of Boolean algebras, we prove that every $\mathbf{B} \in \mathsf{Alg}(\mathrm{PWK})$ is a subalgebra of a power of $\mathbf{WK}$, which entails the following:

Theorem D. $\mathsf{Alg}(\mathrm{PWK})$ *is the quasivariety generated by* $\mathbf{WK}$.

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