On Paraconsistent Weak Kleene Logic and Involutive Bisemilattices - Part I

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Extended abstract

In his *Introduction to Metamathematics* [13, § 64], S.C. Kleene distinguishes between a "strong sense" and a "weak sense" of propositional connectives when partially defined predicates are present. Each of these meanings is made explicit via certain 3-valued truth tables, which have become widely known as *strong Kleene tables* and *weak Kleene tables*, respectively. If the elements of the base set are labelled as $0, \frac{1}{2}, 1$, the strong tables for conjunction, disjunction and negation are given by $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$, $\neg a = 1 - a$. The weak tables for the same connectives, on the other hand, are given by:

$\wedge$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	1	$\frac{1}{2}$	1

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

Each set of tables naturally gives rise to two options for building a many-valued logic, depending on whether we choose to consider only 1 as a designated value, or 1 together with the "middle" value $\frac{1}{2}$. Thus, there are four logics in the Kleene family:

- Strong Kleene logic [13, § 64], given by the strong Kleene tables with 1 as a designated value;
- The Logic of Paradox, LP [17], given by the strong Kleene tables with $1, \frac{1}{2}$ as designated values;
- Bochvar's logic [4], given by the weak Kleene tables with 1 as a designated value;

- Paraconsistent Weak Kleene logic, PWK [12, 19], given by the weak Kleene tables with $1, \frac{1}{2}$ as designated values.

The first three logics have all but gone unnoticed by mathematicians, philosophers, and computer scientists. Strong Kleene logic has applications in artificial intelligence as a model of partial information [1] and nonmonotonic reasoning [22], and in philosophy as a bedrock logic for Kripke's theory of truth and other related proposals [10]; the theory of *Kleene algebras*, moreover, has stirred a considerable amount of interest in general algebra [14]. LP has been fervently supported by Graham Priest in the context of a dialetheic approach to the truth-theoretical and set-theoretical paradoxes, and has enjoyed an enduring popularity that made it the object of intense study both on the proof-theoretical and on the semantical level [18]. And even Bochvar's logic, while not the biggest game in the 3-valued town, is still touched on in several papers and books (see e.g. [3, Ch. 5]).

In terms of sheer impact, PWK is the "ugly duckling" in the family of Kleene logics. Essentially introduced by Halldén [12] and, in a completely independent way, by Prior [19], it is often passed over in silence in the main reviews on finite-valued logics. Most of the extant literature concerns the philosophical interpretation of the third value [2, 5, 9, 12, 23] and a discussion of the so-called *contamination principle* (any sentence containing a subsentence evaluated at $\frac{1}{2}$ is itself evaluated at $\frac{1}{2}$), as well as proof systems of various kinds [2, 7, 8, 11]. An important study on PWK as a consequence relation is [6], to be analysed later in this paper. It has also been noticed early on that the $(2, 2)$-reduct of the 3-element algebra **WK** defined by the weak Kleene tables is an instance of a *distributive bisemilattice*, a notion on which there is a burgeoning literature — actually, the variety of distributive bisemilattices is generated by this reduct. Yet, despite this intriguing connection to algebra, virtually no paper has viewed PWK in the perspective of Algebraic Logic. This makes a sharp contrast with LP, which has been thoroughly studied under this aspect [20, 21]. A partial exception is [11], but a careful assessment of the results in this paper is made difficult by issues with the similarity type of the algebras and logics it considers, and by the authors' failure to adopt the language and concepts of mainstream Abstract Algebraic Logic (AAL).

The aim of this work, which is split into two parts, is to give a contribution towards filling this gap, so as to surmise that the ugly duckling might actually be a gorgeous swan. Firstly, we give a Hilbert-style system for PWK. Actually, we prove the following:

Theorem A. *Given a Hilbert system* $\langle AX, \mathrm{MP}\rangle$ *for* CL*, where* AX *is a set of axioms and* MP *is the only rule, the Hilbert system* $\langle AX, \mathrm{RMP}\rangle$ *is a Hilbert system* *of* PWK*, where* RMP *is the* Restricted Modus Ponens *given by*

$$\text{[RMP]}\ \frac{\alpha \qquad \alpha \to \beta}{\beta} \qquad \textit{provided that } \mathrm{var}(\alpha) \subseteq \mathrm{var}(\beta).$$

Next, we introduce some algebraic structures for PWK, called *involutive bisemilattices*, which are algebras $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ such that $\langle A, \wedge, 0\rangle$ and $\langle A, \vee, 1\rangle$ are a meet and a join semilattices with lower and upper bound, respectively, and $\neg$ is an idempotent operation, satisfying the De Morgan identities, and moreover the equation:

$$x \wedge (\neg x \vee y) \approx x \wedge y.$$

Among other results, we show that involutive bisemilattices are always distributive as bisemilattices and that **WK** generates the variety $\mathcal{IBSL}$ of involutive bisemilattices. More in detail, we prove the following:

Theorem B. *The only nontrivial subdirectly irreducible bisemilattices are* **WK***, the 2-element semilattice* $\mathbf{S}_2$*, and the 2-element Boolean algebra* $\mathbf{B}_2$*, up to isomorphism.*

Finally, we use the algebraic construction of Płonka sums, which was introduced in [15, 16], and prove the following representation theorem for involutive bisemilattices:

Theorem C. *Every involutive bisemilattice is representable as the Płonka sum over a direct system of Boolean algebras.*

As a consequence, we obtain that the equation satisfied by all the involutive bisemilattices are exactly the regular equations satisfied by all the Boolean algebras. We then axiomatise relative to $\mathcal{IBSL}$ its nontrivial subvarieties, namely, Boolean algebras and lower-bounded semilattices. In the second part, we study PWK by recourse to the toolbox of Abstract Algebraic Logic.

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