Expressivity of many-valued coalgebraic logics

Marta Bílková
(joint work with Matěj Dostál)

ICS ASCR in Prague

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Expressivity of modal languages and its limits

Classical modal logics:

- The (finitary) $\Box, \Diamond$ modal logic over image finite Kripke frames is expressive for bisimilarity. (Hennessy-Milner property)
- From Kripke frames to coalgebras (in Set): we have a generic definition of behavioral equivalence, (sometimes captured by bisimilarity), and general methods of proving expressivity.
- Limits of expressivity: depends on the coalgebra functor, the propositional language (arity of conjunction and disjunction), the kind of modalities we allow, and their arities.
  There is a way how to create expressive languages for a large class of coalgebra functors.

Many-valued modal logics:

- The $\Box, \Diamond$ modal logic over image finite Kripke frames with 2-valued accessibility relation and $\mathscr{V}$-valued valuations is not always expressive for bisimilarity (especially if we want to avoid constants for elements of $\mathscr{V}$).
- There is a full (algebraic) characterization in terms of those MTL chains $\mathscr{V}$ for which it is expressive.
  Again, already the propositional logic matters.
- Beyond this (crisp accessibility relation, MTL chains, $\Box, \Diamond$ based logics) not much has been known.

What about coalgebraic generalisations of many-valued modal logics?

Coalgebras in Set

Given an endofunctor $T : \mathrm{Set} \longrightarrow \mathrm{Set}$, a T-coalgebra is a morphism

$$c : X \longrightarrow TX$$

in Set.

Given two T-coalgebras $c : X \longrightarrow TX$ and $d : Y \longrightarrow TY$, the morphism $h : X \longrightarrow Y$ is a homomorphism of T-coalgebras if the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;h\;} & Y \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\ TX & \xrightarrow[\;Th\;]{} & TY \end{array}$$

commutes.

Coalgebras in Set - examples

- If T is the powerset functor P, we obtain Kripke frames. If T is the finitary powerset functor P_ω, we obtain image finite Kripke frames.
- coalgebras for the $\mathscr{V}$-valued powerset functor $[\mathrm{Id}, \mathscr{V}]$ are many-valued Kripke frames, if restricted to those with finite support, they are image finite many-valued Kripke frames.
- coalgebras for the probabilistic distribution functor D_ω are probabilistic Kripke frames
- …

Coalgebras in Set - examples

- coalgebras for the functor $\mathrm{Id} \times \mathrm{Id}$ are binary trees
- coalgebras for the functor $A \times \mathrm{Id}$ are streams over alphabet A
- coalgebras for the functor Id^A are labelled transition systems
- coalgebras for the functor $2 \times \mathrm{Id}^A$ are deterministic automata with input alphabet A
- coalgebras for the functor $2 \times (P\mathrm{Id})^A$ are non-deterministic automata
- coalgebras for the multiset functor are directed weighted graphs
- ...

Propositional part of the logics

Since all our examples come mostly from fuzzy logics, we restrict ourselves to propositional logics extending FL_{ew}. We also fix a commutative integral residuated lattice $\mathscr{V}$, and a countable set of propositional variables At.

$$a := p \mid \top \mid \bot \mid a \wedge b \mid a \vee b \mid a \longrightarrow b \mid a \,\&\, b,$$

Given a coalgebra $c : X \longrightarrow TX$ and a valuation of atoms

$$\|.\|_c : At \longrightarrow [X, \mathscr{V}]$$

the semantics $\| * (a_1, \ldots, a_n)\|_c$ is computed inductively for each n-ary connective $*$, as

$$X \xrightarrow{\overrightarrow{\|a\|_c}} \mathscr{V}^n \xrightarrow{*_{\mathscr{V}}} \mathscr{V}.$$

The semantics of the language can be seen as a local $\mathscr{V}$-valued relation

$$x \Vdash_c a \;=\; \|a\|_c(x).$$

Bisimulations

A relation $B \subseteq X \times Y$ is a bisimulation between $c : X \longrightarrow TX$ and $d : Y \longrightarrow TY$ iff there is a coalgebra structure z on B which makes the projections into coalgebra morphisms:

$$\begin{array}{ccccc} X & \xleftarrow{p_0} & B & \xrightarrow{p_1} & Y \\ \downarrow c & & \downarrow z & & \downarrow d \\ TX & \xleftarrow[Tp_0]{} & TB & \xrightarrow[Tp_1]{} & TY \end{array}$$

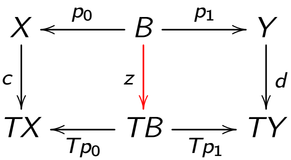
Equivalently (if T preserves weak pullbacks), using *relation lifting*, B is a bisimulation if

$$\overline{T}(B)(c(x), d(y)) \text{ iff } \exists z \in TB(c(x) = (Tp_0)(z) \,\&\, (Tp_1)(z) = d(y)).$$

Remark: If T preserves weak pullbacks, *behavioral equivalence* and *bisimilarity* coincide.

Bisimulations

A relation $B \subseteq X \times Y$ is a bisimulation between $c : X \longrightarrow TX$ and $d : Y \longrightarrow TY$ iff there is a coalgebra structure z on B which makes the projections into coalgebra morphisms:



Two states $x \in X$ and $y \in Y$ are T-bisimilar if there exists a T-bisimulation B such that $B(x,y)$ holds, and moreover the atomic harmony,

$$x' \Vdash_c p = y' \Vdash_d p,$$

holds for all atoms $p \in At$, and all $x' B y'$.

Examples - many-valued Kripke models

A P-bisimulation is a relation $B \subseteq X \times Y$ satisfying: xBy implies

1. $x \Vdash_c p = y \Vdash_d p$ for all atoms,
2. $\forall x' \in c(x)\, \exists y'(y' \in d(y)\ \&\ x'By')$ and
   $\forall y' \in d(y)\, \exists x'(x' \in c(x)\ \&\ x'By')$.

A $P^{\mathscr{V}}$-bisimulation is a relation $B \subseteq X \times Y$ satisfying: xBy implies

1. $x \Vdash_c p = y \Vdash_d p$ for all atoms,
2. $c(x)(x') \leq \bigvee_{y' : x'By'} d(y)(y')$ and $d(y)(y') \leq \bigvee_{x' : x'By'} c(x)(x')$.

Many valued predicate liftings

we define $\mathscr{V}$-valued n-ary *predicate liftings* to be maps:

$$\heartsuit_X : [X, \mathscr{V}^n] \longrightarrow [TX, \mathscr{V}],$$

natural in X.

By an easy observation, n-ary predicate liftings are essentially the same things as the maps

$$\heartsuit : T\mathscr{V}^n \longrightarrow \mathscr{V},$$

which we will call the n-ary modalities:

$$\heartsuit_X(Q) = \heartsuit(TQ) : TX \longrightarrow \mathscr{V} \quad \text{and} \quad \heartsuit = \heartsuit_{\mathscr{V}^n}(\mathrm{id}_{\mathscr{V}^n}).$$

Modal language

We the propositional language with a set Λ of modalities (possibly all). Whenever $a_1, \ldots, a_n$ are formulas and $\heartsuit$ is an n-ary modality, $\heartsuit(a_1, \ldots, a_n)$ is a formula. $\mathscr{L}(\Lambda)$ denotes the resulting modal language.

On a coalgebra $c : X \longrightarrow TX$ with valuations $\|a_i\|_c : X \longrightarrow \mathscr{V}$ the formula $\heartsuit(a_1, \ldots, a_n)$ is interpreted as follows:

$$X \xrightarrow{c} TX \xrightarrow{T\overrightarrow{\|a\|}} T\mathscr{V}^n \xrightarrow{\heartsuit} \mathscr{V}$$

Lemma (Adequacy)

The modal language $\mathscr{L}(\Lambda)$ for Λ a set of predicate liftings (modalities) for T is invariant under bisimilarity.

Boxes

1. The boolean semantics of $\Box : P2 \longrightarrow 2$ is the meet $\bigwedge$:

$$X \xrightarrow{c} PX \xrightarrow{P\|a\|} P(2) \xrightarrow{\bigwedge} 2$$

for any x the result is 1 iff $c(x) = \emptyset$ or $\forall y \in c(x)\ y \Vdash a$.

Boxes

1. Many-valued semantics of $\Box : P\mathscr{V} \longrightarrow \mathscr{V}$ is also the meet, now computed in $\mathscr{V}$:

$$X \xrightarrow{c} PX \xrightarrow{P\|a\|} P(\mathscr{V}) \xrightarrow{\bigwedge} \mathscr{V}$$

for any x the result is $c(x) \Vdash \Box a = \bigwedge_{y \in c(x)} y \Vdash a$.

Boxes

1. Many-valued semantics of $\Box : [\mathscr{V}, \mathscr{V}] \longrightarrow \mathscr{V}$ is given by a mapping:

$$\sigma \mapsto \bigwedge_{v \in \mathscr{V}} (\sigma(v) \longrightarrow v).$$

as

$$X \xrightarrow{c} [X, \mathscr{V}] \xrightarrow{P^{\mathscr{V}} \|a\|} [\mathscr{V}, \mathscr{V}] \xrightarrow{\Box} \mathscr{V}$$

for any x the result is $c(x) \Vdash \Box a = \bigwedge_y (c(x)(y) \longrightarrow y \Vdash a)$.

Separating sets of modalities

A set Λ of predicate liftings for a finitary functor T is called separating iff

$$(\heartsuit^\flat_X : TX \longrightarrow [[X, \mathscr{V}^n], \mathscr{V}])_{n<\omega, \heartsuit\in\Lambda}$$

is jointly injective for all X.

This means that for each $t \neq t'$ in TX, there are some n, $\sigma : X \longrightarrow \mathscr{V}^n$ and n-ary $\heartsuit \in \Lambda$, such that (in $\mathscr{V}$)

$$\heartsuit(T\sigma)(t) \neq \heartsuit(T\sigma)(t').$$

Every finitary functor admits a separating set of predicate liftings. Namely, the set of all n-ary liftings is separating.

Separating sets of modalities

The condition above:
for each $t \neq t'$ in TX, there are some n, $\sigma : X \longrightarrow \mathscr{V}^n$, and n-ary $\heartsuit \in \Lambda$, such that (in $\mathscr{V}$)

$$\heartsuit(T\sigma)(t) \neq \heartsuit(T\sigma)(t').$$

is equivalent to the condition:
for each n and $t \neq t'$ in $T\mathscr{V}^n$, there are some m, $f : \mathscr{V}^n \longrightarrow \mathscr{V}^m$, and m-ary $\heartsuit \in \Lambda$, such that

$$\heartsuit(Tf)(t) \neq \heartsuit(Tf)(t').$$

Consider unary predicate liftings for simplicity: those $f : \mathscr{V}^n \longrightarrow \mathscr{V}$ include truth-functions defined by formulas in the propositional language.

Expresivity

We call a function $f : \mathscr{V}^n \longrightarrow \mathscr{V}$ expressible, if there is a term σ in n variables in the language of $\mathscr{V}$, such that

$$\sigma[x_1, \dots, x_n / v_1, \dots, v_n] = f(v_1, \dots, v_n).$$

Λ is then called $\mathscr{V}$-separating, if the collection of expressible functions separates values in $T\mathscr{V}^n$.

Theorem (Expressivity)

Let T be finitary, w.p.p., and Λ a $\mathscr{V}$-separating set of predicate liftings. Then $\mathscr{L}(\Lambda)$ is expressive for bisimilarity.

Examples

- Each $f : 2^n \longrightarrow 2$ is expressible in the boolean language (because propositional logic is functionally complete).
- for $T = P_\omega$, $\Lambda = \{\Box, \Diamond\}$ and $\mathscr{V}$ a complete MTL chain, the condition of Λ being $\mathscr{V}$-separating is equivalent to the one given by Metcalfe and Martí [Theorem 3.5] in terms of distinguishing formula property.
- If constants for all elements of $\mathscr{V}$ were included in the propositional language, then for $T = P_\omega$ the set $\{\Box, \Diamond\}$ is always $\mathscr{V}$-separating.

G. Metcalfe and M. Marti. *A Hennessy-Milner property for many-valued modal logics.* In Advances in Modal Logic, volume 10, pages 407–420. 2014.

Examples

- Modal logic of P_ω coalgebras based on a $\mathscr{V}$-separating Λ is expressive for bisimilarity.

  This covers results of Metcalfe and Martı. In particular, modal logic of $\{\Box, \Diamond\}$ for $\mathscr{V}$ being an MV algebra, $\mathscr{V} = 2$, or $\mathscr{V} = G_3$ is expressive. The condition on $\mathscr{V}$ in this case is not only sufficient, but also necessary.
- Applying the theory above to a few particular examples, we can moreover show
  - an expressive language for P_ω coalgebras and $\mathscr{V}$ a Gödel chain bigger then 3; and that it cannot be based on unary modalities,
  - the results about expressivity of Łukasziewicz's logics with box and diamond extend to the $\mathscr{V}$-valued finitary powerset functor, and
  - an expressive language for D_ω coalgebras based on a finite set of modalities and the standard Łukasziewicz algebra.

Gödel modal logics

Let P_ω be the coalgebra functor and $\mathscr{V}$ a Gödel chain G_4 with 4 elements, say $0 < u < v < 1$. In this case, $\& = \wedge$ and $a \longrightarrow b = 1$ if $a \leq b$ and $a \longrightarrow b = b$ else, $\bot = 0$ and $\top = 1$.

There is no $\mathscr{V}$-separating set of unary modalities. Therefore there is in general no monadic Gödel modal logic expressive for bisimilarity.

An expressive polyadic modal logic for MTL chains can be defined. The idea is to use the lexicographical linear order on $\mathscr{V}^n$, and define countably many modalities that pick from a finite subset of $\mathscr{V}^n$ the j-th projection of the i-th n-tuple in the set.

Łukasziewicz logics

Let $T = P_\omega^{\mathscr{V}}$. Let $\mathscr{V}$ be the standard Łukasziewicz algebra $[0,1]_{Ł}$, with

$$\begin{aligned} a \,\&\, b &= max\{0, a+b-1\} \\ a \longrightarrow b &= min\{1, 1-a+b\} \\ \neg a = (a \longrightarrow 0) &= 1-a \end{aligned}$$

In particular $a \,\&\, 0 = 0$ for each a, and & distributes over $\vee$. A unary modality $\Diamond : [\mathscr{V}, \mathscr{V}] \longrightarrow \mathscr{V}$ is given, for a $t : \mathscr{V} \longrightarrow \mathscr{V}$ with finite support, by

$$\Diamond t = \bigvee_{u \in \mathscr{V}} (t(u) \,\&\, u).$$

$\Lambda = \{\Diamond\}$ is $\mathscr{V}$-separating. Therefore the modal logics of image finite many-valued Kripke frames based on $\Diamond$ (or $\Box$ or both) and $\mathscr{V}$ the standard or finite Łukasziewicz chain are expressive for bisimilarity.

Łukasziewicz probabilistic logic

Let $T = D_\omega$ (probabilistic Kripke frames). Note that the 2-valued logic for the same coalgebras needs to contain infinitely many modalities.

In $[0,1]_{Ł}$, the truncated sum is definable:

$$a \oplus b = \neg(\neg a \,\&\, \neg b) = max\{1, a+b\}.$$

Consider a unary modality $\diamondsuit : [\mathscr{V}, [0,1]] \longrightarrow \mathscr{V}$ given, for a $t : \mathscr{V} \longrightarrow [0,1]$ with finite support, by

$$\diamondsuit(t) = \bigoplus_{u \in \mathscr{V}} (t(u) \cdot u).$$

$\Lambda = \{\diamondsuit\}$ is $\mathscr{V}$-separating. Therefore the modal logic of image finite probabilistic Kripke frames based on a single modality $\diamondsuit$ and the standard Łukasziewicz algebra is expressive for bisimilarity[1].

[1] $\diamondsuit$ is rather a many-valued then probabilistic modality: a formula $\diamondsuit a$ does not express the probability of a, but rather the truth-value of an expression "probably a".