Expressivity of Many-Valued Coalgebraic Logics*

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Abstract theory of coalgebras has recently became one of the most important bridges connecting modal logic and computer science: from a logician's point of view it provides techniques and a new level of generality for studying various modal logics, while from a computer-scientist's point of view it provides a general framework for designing expressive modal languages describing behavior of abstract transition systems modeled as coalgebras. It is then natural to ask what benefits a coalgebraic approach brings to study of many-valued modal logics: from a logicians' point of view we can generalize logics of many-valued Kripke-style relational semantics to the coalgebraic level, offering in particular a new perspective to the question what is the minimal modal logic over a given residuated lattice $\mathcal{V}$, in which valuations and the accessibility relation take values (cf. [2]). From a computer-scientist's point of view it generalizes coalgebraic logics to the many-valued setting, allowing for a many-valued observable phenomena to be captured by modal languages with many-valued semantics. A notion of behavioral equivalence is central in studying coalgebras, and, in case the coalgebra functor preserves weak pullbacks, it coincides with bisimilarity, a central notion in model theory of modal logics. In particular, at least for systems with a finitary type of branching, we want modal languages expressive for bisimilarity, i.e., satisfying the Hennessy-Milner property.

There are (at least) two general approaches of designing expressive logical languages for finitary coalgebras, both parametric in the coalgebra functor. We adopt the one based on a *logical connection* [1, 3] between modal algebras and coalgebras. It provides one with an abstract machinery producing, for a given coalgebra functor T, a language of *all* available modalities, or equivalently of all *predicate liftings*. Languages based on predicate liftings (for classical coalgebraic logics) were developed and their expressivity investigated by Pattinson in [6, 5] and further by Schröder in [7], where a sufficient condition on a subset of the set of all predicate liftings (namely being separating) is given to ensure that the resulting logic is expressive for behavioral equivalence. We apply this approach in a many-valued setting. Our goal is, given a separating sets of many-valued predicate

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liftings, to give sufficient and necessary conditions on the algebra of truth values for the resulting language being expressive for bisimilarity. In this respect, we are generalising the limitative results of Metcalfe and Martí [4] providing a sufficient and necessary condition on the algebra of truth values $\mathcal{V}$ ensuring the Hennessy-Milner property for the basic modal language with box and diamond over crisp $\mathcal{V}$-valued image-finite Kripke frames where $\mathcal{V}$ is a complete MTL-chain. Roughly speaking, the condition they provide says that we can distinguish truth values in $\mathcal{V}$ with propositional formulas, so also the purely propositional part of the language matters. If we want to avoid including constants for all truth values in the language, the condition rules out many interesting logics: for $\mathcal{V}$ being a complete BL-chain with finite universe or $[0, 1]$, this yields expressivity if and *only if* $\mathcal{V}$ is a MV-chain or the ordinal sum of two (hoop reducts of) MV-chains.

Many-valued predicate liftings. We consider many-valued logical connections and adapt the notion of a separating set of predicate liftings to the many-valued setting, including now also a condition on the algebra $\mathcal{V}$ (of which the condition given in [4] is a special case for the coalgebra functor being the finitary powerset functor and the only modalities being box and diamond). We prove that, given a separating set of modalities and the additional condition on $\mathcal{V}$, the resulting logic is expressive for bisimilarity. We illustrate the theorem with various examples, namely some not covered by [4].

Technically speaking, given a finitary coalgebra functor T and a complete residuated lattice $\mathcal{V}$, n-ary *modalities* arising from a logical connection between T-coalgebras and residuated lattices induced by $\mathcal{V}$ are semantically the maps:

$$\heartsuit : T\mathcal{V}^n \to \mathcal{V}$$

Modalities are in one-to-one correspondence with n-ary *predicate liftings* $\hat{\heartsuit}_X : [X, \mathcal{V}^n] \to [TX, \mathcal{V}]$, natural in X. A set Λ of predicate liftings is called *separating* iff the transpose

$$(\hat{\heartsuit}^\flat_X : TX \to [[X, \mathcal{V}^n], \mathcal{V}])_{\heartsuit \in \Lambda}$$

is *jointly injective* for all X. For a finitary functor, the set of all the predicate liftings it induces is always separating. A separating Λ can be equivalently characterized as satisfying the following:

$$t \neq t' \text{ in } T\mathcal{V}^n \text{ implies } \exists \sigma : \mathcal{V}^n \to \mathcal{V}^n, \heartsuit \in \Lambda \text{ such that } \heartsuit(T\sigma)(t) \neq \heartsuit(T\sigma)(t').$$

The condition says that we could distinguish values in $T\mathcal{V}^n$ using the modalities in Λ, if σ was expressible in the propositional part of the language: we call

$\sigma : \mathcal{V}^n \to \mathcal{V}^n$ *expressible*, if there are n terms $\sigma_1, \ldots, \sigma_n$ in n variables in the language of $\mathcal{V}$, such that:

$$\sigma(v_1, \ldots, v_n) = (\sigma_1(v_1, \ldots, v_n), \ldots, \sigma_n(v_1, \ldots, v_n)).$$

The condition we put on $\mathcal{V}$ is therefore the following (for fixed T and Λ):

Condition. *$t \neq t'$ in $T\mathcal{V}^n$ implies there exists $\sigma : \mathcal{V}^n \to \mathcal{V}^n$ expressible, and $\heartsuit \in \Lambda$ such that $\heartsuit(T\sigma)(t) \neq \heartsuit(T\sigma)(t')$.*

Now we can prove the expressivity for the modal language induced by Λ:

Theorem (Expressivity)**.** *Let T be finitary, Λ a separating set of predicate liftings, and $\mathcal{V}$ satisfies the condition (w.r.t. T and Λ). Then $\mathcal{L}(\Lambda)$ is expressive for bisimilarity.*
For a fixed T and Λ, the condition for $\mathcal{V}$ is sufficient and necessary.

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