Epimorphism surjectivity and the Beth definability property

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A morphism h in a category K is called a (K–) *epimorphism* provided that, for any two K–morphisms f, g from the co-domain of h to a single object,

$$\text{if } f \circ h = g \circ h, \text{ then } f = g.$$

A class of algebras K has the *ES property* when epimorphisms are surjective in it. The ES property is algebraically natural, but our main motivation comes from logic. The algebraic counterpart K of an algebraizable logic $\vdash$ is a prevariety, i.e., a class of similar algebras, closed under isomorphisms, subalgebras and direct products [1, 2]. The logical meaning of the ES property is the following [1, Thm. 3.17] (see also [7, 8]):

K has the ES property iff $\vdash$ has the *infinite Beth definability property*.

The latter signifies that, in $\vdash$, whenever a set Z of variables is defined *implicitly* in terms of a disjoint set X of variables by means of some set Γ of formulas over $X \cup Z$, then Γ also defines Z *explicitly* in terms of X. The *finite Beth property* makes the same demand, but only when Z is finite. An algebraizable logic $\vdash$ has the finite Beth property iff its algebraic counterpart K has the following 'weak' ES property [1, 6]:

if $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is an epimorphism in K and $\boldsymbol{B}$ is generated by $h[A] \cup C$ for some finite set $C \subseteq B$, then h is onto.

Blok and Hoogland conjectured in [1, p. 76] that the infinite Beth property is not equivalent to its finite version. We shall confirm their conjecture by exhibiting a locally finite variety of Heyting algebras with the weak ES property but not the ES property. In logical terms, we describe a locally tabular intermediate logic in which the finite Beth property holds and the infinite Beth property fails. Recall that the concept of depth and width of a Heyting algebra are defined in terms of the poset of its prime filters [9, 13]. Our counterexample can be used to obtain the following:

Theorem 1. *There is a continuum of varieties of Heyting algebras of width 2 with the weak ES property, but without the ES property.*

In fact the weak ES property holds in every variety of Heyting algebras [10]. It is therefore sensible to ask which varieties of Heyting algebras have the ES property. Using Esakia duality, we prove the following:

Theorem 2. *Every variety of Heyting algebras of finite depth has the ES property.*

At depth 3, this already supplies a continuum of varieties of Heyting algebras with the ES property. Since Maksimova showed that there are finitely many varieties of Heyting algebras with the *strong ES property*, this provides uncountably many varieties of Heyting algebras with the ES property but without its strong version [11, 12]. Another consequence of the theorem is that epimorphisms are surjective in every *finitely generated* variety of Heyting algebras.

Finally we exploit the category equivalences in [4, 5], to transfer our results to some non-integral varieties of residuated lattices [3]. In particular, we show that epimorphisms are surjective in all varieties consisting of Sugihara monoids or positive Sugihara monoids. These are the algebraic counterpart respectively of the axiomatic extensions of the relevance logic $\mathbf{RM^t}$ and of its negation-less fragment.

References

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