IT4I

Towards Predicate Fuzzy Partial Logic

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OP Výzkum a vývoj pro inovace

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Fuzzy partial propositional logic

Fuzzy partial logic = fuzzy logic admitting undefined truth of propositions

Admissible underlying fuzzy logics: any implicative extension L of $\mathrm{MTL}_\triangle$

(eg, $Ł_\triangle$, ŁΠ, Bool, ...)

Intended algebras: $\boldsymbol{L}_* = \boldsymbol{L} \cup \{*\}$ for an L-algebra $\boldsymbol{L}$

Connectives of $\boldsymbol{L}$ can be extended to $\boldsymbol{L}_*$ in several meaningful ways

(variant connectives available in $\boldsymbol{L}_*$, the choice left to the user)

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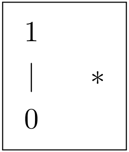

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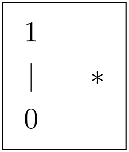

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Propositional connectives of fuzzy partial logic

(a) Bochvar-style connectives ($*$ is the annihilator):

x	ux
α	$u\alpha$
$*$	$*$

c	β	$*$
α	$\alpha\, c\, \beta$	$*$
$*$	$*$	$*$

(and analogously for higher arities)

(b) Sobociński-style connectives ($*$ is neutral):

$\hat{\wedge}$	β	$*$
α	$\alpha \wedge \beta$	α
$*$	β	$*$

$\hat{\vee}$	β	$*$
α	$\alpha \vee \beta$	α
$*$	β	$*$

$\hat{\&}$	β	$*$
α	$\alpha \mathbin{\&} \beta$	α
$*$	β	$*$

$\hat{\to}$	β	$*$
α	$\alpha \to \beta$	$\neg\alpha$
$*$	β	$*$

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$\hat{\&}$	β	$*$
α	$\alpha \,\&\, \beta$	α
$*$	β	$*$

$\hat{\rightarrow}$	β	$*$
α	$\alpha \rightarrow \beta$	$\neg\alpha$
$*$	β	$*$

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Propositional connectives of fuzzy partial logic

(c) Kleene-style connectives ($\boldsymbol{L}$-annihilators, otherwise Bochvar)

$\bar{\wedge}, \bar{\&}$	0	β	$*$
0	0	0	0
α	0	$\alpha \wedge/\& \ \beta$	$*$
$*$	0	$*$	$*$

$\bar{\vee}$	δ	1	$*$
γ	$\gamma \vee \delta$	1	$*$
1	1	1	1
$*$	$*$	1	$*$

$\bar{\rightarrow}$	δ	1	$*$
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(d) Bochvar-external–style, McCarthy-style, interval-valued–style, . . .

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Auxiliary unary connectives:

x	$!x$	$\downarrow x$	$\uparrow x$
α	1	α	α
$*$	0	0	1

Crisp comparison of truth values:

$\equiv$	β	$*$
α	$\triangle(\alpha \leftrightarrow \beta)$	0
$*$	0	1

$\sqsubseteq$	β	$*$
α	$\triangle(\alpha \leftrightarrow \beta)$	0
$*$	1	1

$\trianglelefteq$	β	$*$
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$\trianglelefteq$	β	$*$
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Fuzzy partial propositional logic

- Primitive language: $\mathsf{L} \cup \{*, !, \bar{\wedge}\}$ (other connectives definable)
- Tautologicity and entailment: wrt the truth degree 1
- Axiomatic system: $\Gamma \vdash \uparrow\varphi$ (for finite $\Gamma \vdash_{\mathsf{L}} \varphi$) + 4 rules + 10 axioms
- Deduction theorem:

  $$\Gamma, \varphi \vdash \psi \quad \text{iff} \quad \Gamma \vdash (\varphi \equiv 1) \to (\psi \equiv 1) \quad \text{iff} \quad \Gamma \vdash \downarrow\triangle\varphi \to \downarrow\psi$$

- Completeness: wrt $\boldsymbol{L}_*$ for $\boldsymbol{L}$ general/linear/standard L-algebras (the logic is Rasiowa-implicative wrt $\trianglelefteq$)
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Partial fuzzy logic of the first order

First-order models:

- Universe $U \neq \emptyset$
- Function symbols: $F\colon U^n \to U$
- Predicate symbols: $P\colon U^n \to \boldsymbol{L}_*$

NB: We only consider undefined truth values, not undefined individuals

= future work

All definitions as usual, we only need to specify the semantics of quantifiers

The $\boldsymbol{L}$-valued quantifiers $\forall$,$\exists$ can be extended to $\boldsymbol{L}_*$ in several meaningful ways

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Partial fuzzy quantifiers

(a) Bochvar-style quantifiers

$$\|(\forall x)\varphi\| = \begin{cases} * & \text{if } \|\varphi(a)\| = * \text{ for some } a \in U \\ \inf_{a \in U} \|\varphi(a)\| & \text{otherwise} \end{cases}$$

$$\|(\exists x)\varphi\| = \begin{cases} * & \text{if } \|\varphi(a)\| = * \text{ for some } a \in U \\ \sup_{a \in U} \|\varphi(a)\| & \text{otherwise} \end{cases}$$

$\forall, \exists$ = the primitive quantifiers of Ł$\forall^*$

Not very useful themselves, but other meaningful quantifiers are definable by means of $\forall, \exists$ and propositional connectives of Ł*

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(b) Sobociński-style quantifiers

$$\|(\hat{\forall}x)\varphi\| = \begin{cases} * & \text{if } \|\varphi(a)\| = * \text{ for all } a \in U \\ \inf_{a\in U} \|{\uparrow}\varphi(a)\| & \text{otherwise} \end{cases}$$

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Definition in terms of $\forall$,$\exists$:

$$(\hat{\forall}x)\varphi \equiv_{\mathsf{df}} (\forall x){\uparrow}\varphi \vee \circledast(\forall x)\neg!\varphi$$

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(c) Kleene-style quantifiers

$$(\bar{\forall} x)\varphi \equiv_{\mathsf{df}} (\forall x)\varphi \mathbin{\bar{\wedge}} (\hat{\forall} x)\varphi$$

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Further meaningful quantifiers arise by combining $\forall, \exists$ with $\uparrow, \downarrow$:

- $(\forall x){\uparrow}\varphi$ Sette universal quantifier
  (undefined instances ignored, void $\mapsto 1$ = "vacuously true")
- $(\forall x){\downarrow}\varphi$ Bochvar-external universal quantifier
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Other combinations are either equivalent to these or not meaningful

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Properties of basic fuzzy partial quantifiers

Observations

- $\varphi \models (\mathsf{Q}x)\varphi$ for Q $\in \{\forall, \hat{\forall}, \bar{\forall}, \exists, \hat{\exists}, \bar{\exists}\}$ and all combinations with $\uparrow, \downarrow$
- $(\mathsf{Q}x)\varphi \models \varphi$ for Q $\in \{\forall, \bar{\forall}\}$ and all combinations with $\downarrow$
- In general, however, $(\hat{\forall}x)\varphi \not\models \varphi$; only $(\hat{\forall}x)\varphi \models {\uparrow}\varphi$
- $\models (\forall x){\downarrow}\varphi \to (\forall x){\uparrow}\varphi$ (Bochvar-external $\forall$ is stronger than Sette), etc

Axiomatization:

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$\Rightarrow$ standard (Rasiowa) quantifier axioms work for L$\forall^*$

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Theory of partial fuzzy sets

PFCT_1 = a theory in multi-sorted first-order Ł∀*

Language:

- Variables for elements $x, y, \ldots$
- Variables for (fuzzy partial) classes $A, B, \ldots$
- Equality predicate $=$ on each sort (crisp, total)
- Membership predicate $\in$ between elements and classes (fuzzy, partial)
- Class terms $\{x \mid \varphi(x)\}$ for each x, φ

Convention: abbreviate $x \in A$ by Ax

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- Variables for (fuzzy partial) classes $A, B, \ldots$
- Equality predicate $=$ on each sort (crisp, total)
- Membership predicate $\in$ between elements and classes (fuzzy, partial)
- Class terms $\{x \mid \varphi(x)\}$ for each x, φ

Convention: abbreviate $x \in A$ by Ax

IRAFM

Theory of partial fuzzy sets

PFCT_1 = a theory in multi-sorted first-order Ł∀*

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Theory of partial fuzzy sets

Intended models:

- L_*-valued membership functions over a given domain of elements
- $\|x \in A\| = \|A\| \left(\|x\|\right)$
- $\|\{x \mid \varphi(x)\}\| : \|x\| \mapsto \|\varphi(x)\|$

Intended models of PFCT_1 are not axiomatizable

Axiomatic approximation (Henkin models):

- Equality: $x = x$ and $x = y \to (\varphi(x) \equiv \varphi(y))$, ditto for classes
- Extensionality: $(\forall x)(Ax \equiv Bx) \to A = B$
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Iterate for PFCT of higher orders, add tuples as in FCT (crisp)

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Notions coming from the theory of partial functions

Definition

$\mathrm{dom}\, A =_{\mathsf{df}} \{x \mid {!Ax}\}$	domain
$A = B \equiv_{\mathsf{df}} (\forall x)(Ax \equiv Bx)$	strong equality
$A \mathrel{\dot{=}} B \equiv_{\mathsf{df}} (\forall x)((Ax \equiv Bx) \vee \neg !Ax \vee \neg !Bx)$	weak equality
$A \mathrel{\hat{=}} B \equiv_{\mathsf{df}} (\forall x)(\downarrow Ax \equiv \downarrow Bx)$	equality on supports
$A \sqsubseteq B \equiv_{\mathsf{df}} (\forall x)(Ax \sqsubseteq Bx)$	subfunction

Observe:

- $=$, $\mathrel{\hat{=}}$ are crisp totally defined equivalence relations
- $\sqsubseteq$ is a crisp totally defined ordering, etc

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Class constants, heights and plinths

Definition

$\emptyset =_{\mathsf{df}} \{x \mid 0\}$	empty class
$\lambda =_{\mathsf{df}} \{x \mid *\}$	totally undefined class
$\mathsf{V} =_{\mathsf{df}} \{x \mid 1\}$	universal class

Definition

$\mathsf{Hgt} A \equiv_{\mathsf{df}} (\exists x) Ax$	$\mathsf{Plt} A \equiv_{\mathsf{df}} (\forall x) Ax$	Bochvar height and plinth
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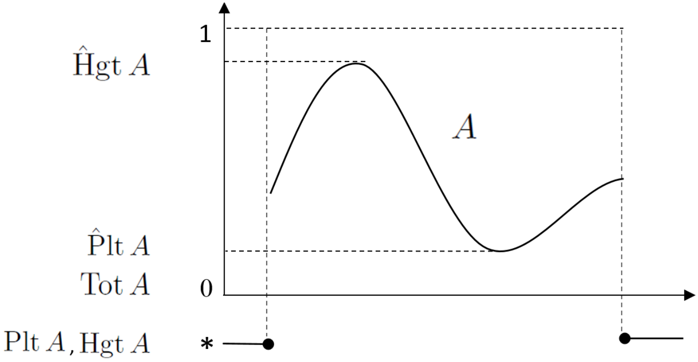

Basic class operations and relations

Definition

$$A \cap B =_{\mathsf{df}} \{x \mid Ax \wedge Bx\}$$ Bochvar min-intersection

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and analogously for $\cup, \doublecap, \ldots$

Observation

$$\mathrm{dom}(A \cap B) = \mathrm{dom}\, A \cap \mathrm{dom}\, B$$

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Reduction to propositional properties

Let $\varphi(p_1, \ldots, p_k)$ be a propositional formula, $Q \in \{\forall, \hat{\forall}, \bar{\forall}, \exists, \hat{\exists}, \bar{\exists}\}$

Definition

$$\mathrm{Op}_\varphi(A_1, \ldots, A_k) =_{\mathrm{df}} \{x \mid \varphi(A_1 x, \ldots, A_k x)\}$$

$$\mathrm{Rel}_\varphi^Q(A_1, \ldots, A_k) \equiv_{\mathrm{df}} (Qx)\varphi(A_1 x, \ldots, A_k x)$$

Examples: $\mathrm{Op}_\triangle$ is $\ker$, $\mathrm{Op}_{\hat{\wedge}}$ is $\hat{\cap}$, $\mathrm{Rel}_\equiv^\forall$ is $=$, etc

Theorem

The following conditions are equivalent:

- $\mathrm{L}^* \models \varphi(\psi_1, \ldots, \psi_n)$
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Reduction to propositional properties

Corollaries

PFCT_1		MTL^*
$\models A \mathbin{\dot\cap} (B \mathbin{\dot\cap} C) = (A \mathbin{\dot\cap} B) \mathbin{\dot\cap} C$	by	$\models p \mathbin{\&} (q \mathbin{\&} r) \equiv (p \mathbin{\&} q) \mathbin{\&} r$
$\not\models \emptyset \subseteq A$	by	$\not\models 0 \to p$
$\models \lambda \sqsubseteq A$	by	$\models (* \equiv p) \vee \neg !*$
$\models \mathrm{dom}(A \mathbin{\hat\cap} B) = \mathrm{dom}\, A \cup \mathrm{dom}\, B$	by	$\models !(p \mathbin{\hat\vee} q) \equiv (!p \vee !q)$, etc.

Further metatheorems of a similar form . . . work in progress

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Representation by total fuzzy classes

Observation

A fuzzy partial set A is determined by:

- a crisp domain $X = \text{dom}\, A$ and
- a (total) fuzzy subset F of X

$\Rightarrow$ Syntactic interpretation FCT $\longleftrightarrow$ PCFT

$$\langle X, F \rangle \mapsto F \cap \text{supp}^* X$$
$$A \mapsto \langle \text{dom}\, A, \Downarrow A \rangle$$

PFCT's relations and operations handle both components $\langle X, F \rangle$ simultaneously, in a variety of predefined manners

Example

$$\langle X_1, F_1 \rangle \cap \langle X_2, F_2 \rangle =_{\text{df}} \langle X_1 \cap X_2, F_1 \cap F_2 \rangle$$
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